The Lefschetz Number of a Map

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November 2021

1 Algebraic Nonsense

(i) **The Definition of Trace**

Let G be a finitely generated free Abelian group (thus, isomorphic to $\mathbb{Z}^N$ for some $N \in \mathbb{N}$) and let $A = (a_{ij})$ be a matrix with coefficients in G. Define the *trace* of the matrix A as

$$\operatorname{tr}(A) = \sum_{i=1}^{N} a_{ii}.$$

It is elementary to verify that for any such $n \times n$ matrices A and B with elements in G, we have the following "commutativity" property:

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

We can uniquely extend this definition to homomorphisms $\varphi : G \to G$. Let A be a matrix which represents φ with respect to some basis of G, and define

$$\operatorname{tr}(\varphi) = \operatorname{tr}(A).$$

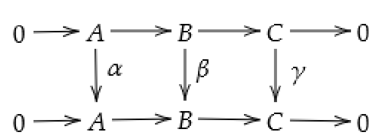
This is well defined since, for other matrix A' which represents φ in another basis of G, it will have the form $A' = BAB^{-1}$ for some invertible change-of-basis matrix B. Thus, by the commutativity property mentioned in the last paragraph,

$$\operatorname{tr}(A') = \operatorname{tr}(BAB^{-1}) = \operatorname{tr}(B^{-1}BA) = \operatorname{tr}(A).$$

If G is finitely generated and Abelian, but not necessarily free, let $T = \{g \in G : \exists\, n \in \mathbb{N}, ng = 0\}$ be the *torsion* subgroup of G. Let $\varphi : G \to G$ be a endomorphism. Then, since φ is a homomorphism, $\varphi(T) \subset T$, so φ induces a homomorphism $\bar{\varphi} : G/T \to G/T$. Since G/T is free, finitely generated, and Abelian, we simply define the trace of φ to be $\operatorname{tr}(\bar{\varphi})$.

(ii) **A Lemma Regarding Trace Additivity**

Let A, B, and C be finitely generated Abelian groups with maps $\iota, \jmath, \alpha, \beta$, and γ such as in the commutative diagram above, such that each square commutes and each horizontal row is exact. Then, $\operatorname{tr}(\beta) = \operatorname{tr}(\alpha) + \operatorname{tr}(\gamma)$.



Proof. First, we reduce to the case where A, B, and C are torsion-free. For any group G, we denote its torsion subgroup by T_G. It is easy to see that forming A/T_A, B/T_B, and $C/\jmath(T_B)$ induces well-defined maps and homomorphisms which make the diagram exact and commute, since that's what they did before. This is the easy part, and it removes the torsion in A and B. None of the traces of the operators α, β and γ have changed.

So, we may assume A and B lack torsion. To eliminate the torsion in C, consider the pre-image $A' = \jmath^{-1}(T_C) \subset B$. Since this sequence is exact, $\iota(A) \subset j^{-1}(T_C)$. Note that, since B is finitely generated and free abelian, there is a basis $\{b_1, b_2, \cdots, b_n\}$ of B and integers $d_1, d_2, \cdots, d_r$ so that $\{d_1c_1, d_2c_2, \cdots d_rc_r\}$ is a basis of $\iota(A)$. Then, it follows that the span of $\{d_1, \cdots, d_r\}$ is A'. If we replace A with A', ι with the inclusion $A' \to B$, α with α' the restriction of β to the subgroup A', C with $C' = C/T_C$, and γ with the induced map $\gamma' : C' \to C'$, then the diagram still commutes and the rows are still exact. The result will then follow from the case of all groups being torsion free if we know that $\mathrm{tr}(\alpha') = \mathrm{tr}(\alpha)$. But, this is trivial since the action of α on A is the same as the action of β on $\iota(A)$ which has the same matrix as the restriction of β to A'.

If all the groups are torsion free, then there exists a basis $\{b_1, b_2, \cdots, b_n\}$ of B so that $\{b_1, \cdots, b_r\}$ is a basis of $\iota(A)$ and $\{\jmath(b_{r+1}), \cdots, \jmath(b_n)\}$ is a basis of C. Then, the form of the matrix representing β is

$$\begin{bmatrix} M_1 & X \\ 0 & M_2 \end{bmatrix},$$

where M_1 is the same as the matrix representing α in the basis $\{\iota^{-1}(b_1), \cdots, \iota^{-1}(b_r)\}$ and M_2 is the same as the matrix representing γ in the induced basis, by commutativity. Thus, $\mathrm{tr}(\beta) = \mathrm{tr}(\alpha) + \mathrm{tr}(\gamma)$.

□

2 The Lefschetz Number

Let X be a space and $f : X \to X$ a continuous map, so that f induces homomorphisms $f_*^n : H_n(X) \to H_n(X)$. Suppose additionally that the homology groups of X vanish for sufficiently high dimensions (e.g., a finite dimensional CW complex). Define the *Lefschetz number* of f to be the alternating sum of traces:

$$L(f) = \sum_{n=0}^{\infty} (-1)^n \,\mathrm{tr}\left(f_*^n\right).$$

We have the following theorem.

Theorem 1 (Lefschetz Fixed Point Theorem)**.** *Suppose X is a finite simplicial complex. If $f : |X| \to |X|$ has no fixed points, then $L(f) = 0$.*

Proof. The main ingredients here are the simplicial approximation theorem (see Hatcher, p. 177) and the algebraic lemma in section 1 above. Put any metric d on X which, say, restricts to the induced Euclidean metric on each open simplex in X. Since X is compact and the continuous function $x \mapsto d(x, f(x))$ is never zero, then there exists a $\varepsilon > 0$ so $d(x, f(x)) > \varepsilon$ for every $x \in X$. Subdivide X into a finer simplicial complex L such that the open star of any simplex of any dimension has diameter less than $\varepsilon/2$. By the simplicial approximation theorem, there is a subdivision K of L and a *simplicial approximation* $g : K \to L$ to f which is, in particular, homotopic to f.

Recall that the defining property of a simplicial approximation is that, for any vertex $v \in X$, $f(\mathrm{St}(v)) \subset \mathrm{st}(g(v))$, where st and St denote the open and closed stars, respectively. In particular, for any simplex σ in X, $f(\sigma)$ is contained in the star of $g(\sigma)$.

But then, for any simplex σ of K, the intersection $g(\sigma) \cap \sigma$ is empty, since for any $x \in \sigma$, σ lies within $\varepsilon/2$ of x and $g(\sigma)$ lies within $\varepsilon/2$ of $f(x)$, since K is a subdivision of L.

Write $H_n(|X|) = H_n$. Since g is homotopic to f, they induce the same homomorphisms $g_*^n : H_n \to H_n$ and consequently $L(g) = L(f)$.

Since g is simplicial, the n-skeleton K^n is taken to the n-skeleton L^n for every n. But then $g(K^n) \subset L^n \subset K^n$, and hence g induces a chain map of the cellular chain complex

$$\cdots \xrightarrow{d_{n+2}} H_{n+1}\left(K^{n+1}, K^n\right) \xrightarrow{d_{n+1}} H_n\left(K^n, K^{n-1}\right) \xrightarrow{d_n} H_{n-1}\left(K^{n-1}, K^{n-2}\right) \xrightarrow{d_{n-1}} \cdots$$

to itself. Henceforth denote $H_n(K^n, K^{n-1}) = \overline{H}_n$. We already proved in class that the homology groups of this complex are isomorphic to H_n. Denoting $Z_n = \ker d_n$ and $B_n = \operatorname{im} d_{n+1}$, we have the following diagrams for each $n = 0, 1, \cdots$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_n & \longrightarrow & Z_n & \longrightarrow & H_n & \longrightarrow & 0 \\ & & \downarrow g_* & & \downarrow g_* & & \downarrow g_* & & \\ 0 & \longrightarrow & B_n & \longrightarrow & Z_n & \longrightarrow & H_n & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_n & \longrightarrow & \overline{H}_n & \longrightarrow & B_{n-1} & \longrightarrow & 0 \\ & & \downarrow g_* & & \downarrow g_* & & \downarrow g_* & & \\ 0 & \longrightarrow & Z_n & \longrightarrow & \overline{H}_n & \longrightarrow & B_{n-1} & \longrightarrow & 0 \end{array}$$

where each horizontal row is exact.

To simplify notation, we denote the trace of the endomorphism induced by g on any Abelian group A in the above diagrams by $\operatorname{tr}(A)$. By the algebraic lemma, we have that

$$\operatorname{tr}(Z_n) = \operatorname{tr}(B_n) + \operatorname{tr}(H_n),$$
$$\operatorname{tr}(\overline{H}_n) = \operatorname{tr}(Z_n) + \operatorname{tr}(B_{n-1}).$$

By substituting the first equation into the second and taking the alternating sum over n, we obtain:

$$\sum_{n=0}^{\infty}(-1)^n \operatorname{tr}(\overline{H}_n) = \left(\sum_{n=0}^{\infty}(-1)^n \operatorname{tr}(H_n)\right) + \sum_{n=0}^{\infty}(-1)^n(\operatorname{tr}(B_n) + \operatorname{tr}(B_{n-1})),$$

and since B_{-1} is the trivial group and the top plus one-level B_N is trivial as well, the sum on the right telescopes and becomes equal to zero. Thus,

$$L(g) = \sum_{n=0}^{\infty}(-1)^n \operatorname{tr}\left(g_* : \overline{H}_n \to \overline{H}_n\right)$$

This is great, because it allows us to compute the Lefschetz number on the level of the cellular chain complex of K, and this has a particularly simple description as covered in class. The groups $\overline{H}_n = H_n(K^n, K^{n-1})$ are free and abelian, with generators in a bijective correspondence to the open n-simplices of K. But, in this basis, the trace of the matrix representing each g_*^n has zeros along the diagonal since $g(\sigma) \cap \sigma = \emptyset$ for any simplex $\sigma \in K$, so $L(f) = L(g) = 0$ □

More generally, if X is a retract of a finite simplicial complex K, then $H_n(X)$ is a direct summand of $H_n(K)$ (Hatcher p. 147) and the projection r_* induced by the retraction r is projection onto that summand. The composition $f \circ r$ has the same fixed points as f, and hence $\operatorname{tr}(f_* \circ r_*) = \operatorname{tr}(f_*)$, so $L(f) = L(f \circ r)$.