SPIN STRUCTURES AND THE SECOND STIEFEL-WHITNEY CLASS

1. Classification of real vector bundles.

Theorem. Every real vector bundle of rank n over a paracompact space B is isomorphic to $f^*\gamma^n$, for some $f : B \to G_n(\mathbb{R}^\infty)$. (Where γ^n is the canonical rank n bundle over $G_n(\mathbb{R}^\infty)$.)

If $f, g : B \to G_n(\mathbb{R}^\infty)$ are homotopic, $f^*\gamma^n$ and $g^*\gamma^n$ are isomorphic; and conversely.

Thus we have a bijection:

$$\phi_B : [B, G_n(\mathbb{R}^\infty)] \leftrightarrow Vect_n(B), \quad \phi_B([f]) = f^*(\gamma_n).$$

($Vect_n(B)$ is the set of isomorphism classes of rank n vector bundles over B.)

References:[Husemoller 3(4.7), 3(5.6), 3(6.2)]; [Milnor-Stasheff 5.6, 5.7].

2. Classification of principal bundles. Let G be a compact Lie group.

For each paracompact space (or smooth manifold) B, let $Prin_G(B)$ denote the set of isomorphism classes of principal G-bundles over B. A principal G bundle $(\omega)EG \to BG$ is a *universal* G-bundle if for each paracompact space X, the 'induced bundle map' ϕ_X is a bijection:

$$\phi_X : [X, BG] \to Prin_G(X), \quad \phi_X(u) = u^*(\omega) \in Prin_G(X).$$

and if:

$$f, g : X \to BG, \text{ then } f^*\omega, g^*\omega \text{ isomorphic } \Leftrightarrow f, g \text{ homotopic }.$$

BG is a 'classifying space' for G.

Theorem. (see [Steenrod.) If $p : E \to B$ is a principal G-bundle and the total space E is *contractible*, then it is a universal G-bundle.

Theorem. (Milnor's join construction.) For any Lie group G there exists a (unique up to homotopy type) classifying space $EG \to BG$, with EG contractible. (See [Husemoller] for a proof.)

Examples:

$$BO_n = G_n(\mathbb{R}^\infty).$$
$$BSO_n = G_n^+(\mathbb{R}^\infty) \text{ (oriented } n\text{-planes)}.$$
$$G = \mathbb{Z}_2 : EG = S^\infty \text{ (contractible)}, BG = P^\infty = K(\mathbb{Z}_2, 1).$$
$$G = S^1 : EG = S^\infty, BG = \mathbb{C}P^\infty = K(\mathbb{Z}, 2).$$

3. Definition of Stiefel-Whitney classes via classifying spaces.

The cohomology ring $H^*(G_n(\mathbb{R}^\infty);\mathbb{Z}_2)$ is a $\mathbb{Z}_2$ polynomial algebra $\mathbb{Z}_2[w_1,\dots w_n]$ freely generated by $w_1,\dots,w_n$, where $w_k \in H^k(G_n(\mathbb{R}^\infty);\mathbb{Z}_2)$. In particular: $H^1(BO_n,\mathbb{Z}_2)\sim\mathbb{Z}_2$, with generator w_1.

[M-S p.81] The correspondence $X\mapsto\mathbb{R}^1\oplus X$ defines an embedding $G_n(\mathbb{R}^m)\to G_{n+1}(\mathbb{R}^1\oplus\mathbb{R}^m)=G_{n+1}(\mathbb{R}^{m+1})$, covered by a bundle map:

$$\epsilon^1\oplus\gamma^n(\mathbb{R}^m)\to\gamma^{n+1}(\mathbb{R}^{m+1}).$$

This direct-limits (as $m\to\infty$) to an embedding $G_n(\mathbb{R}^\infty)\hookrightarrow G_{n+1}(\mathbb{R}^\infty)$ of classifying spaces, covered by a bundle map $\epsilon^1\oplus\gamma^n\to\gamma^{n+1}$ (This corresponds to the inclusion $O(n)\hookrightarrow O(n+1)$.)

[L-M p.380] Setting $w_0=1$, the cohomology class w_{n+1} can be inductively characterized by the fact the kernel of the restriction map $H^*(G_{n+1}(\mathbb{R}^\infty);\mathbb{Z}_2)\to H^*(G_n(\mathbb{R}^\infty);\mathbb{Z}_2)$ is the ideal $\langle w_{n+1}\rangle$.

Now *define* the i^{th} Stiefel-Whitney class of a rank n vector bundle $(\xi)E\to X$ by $w_i(\xi)=f^*w_i\in H^i(X,\mathbb{Z}_2)$, where $f:X\to G_n(\mathbb{R}^\infty)$ is any classifying map for ξ.

Uniqueness for w_1. Note that $H^1(BO_n,\mathbb{Z}_2)\sim\mathbb{Z}_2$, so it has a unique nonzero element $w_1(\gamma^n)$. Thus if an assignment $\bar w_1$ of a cohomology class in $H^1(X;\mathbb{Z}_2)$ to each rank n vector bundle satisfies the conditions:

(i) Naturality under bundle maps;

(ii) $\bar w_1(\gamma^n)$ is the nonzero element in $H^1(BO_n;\mathbb{Z}_2)$,

then it must coincide with the first Stiefel-Whitney class, as defined above.

For oriented vector bundles, a similar definition can be made, based on $BSO_n=G_n^+(\mathbb{R}^\infty)$: the cohomology algebra $H^*(BSO_n,\mathbb{Z}_2)$ is a $\mathbb{Z}_2$ polynomial algebra $\mathbb{Z}_2[y_2,\dots,y_n]$ freely generated by $y_i\in H^i(BSO_n,\mathbb{Z}_2)$, the pullbacks of the w_i under the inclusion $BSO_n\hookrightarrow BO_n$. In particular, we have:

$$H^2(BSO_n,\mathbb{Z}_2)\sim\mathbb{Z}_2$$

[L-M p. 381], and we *define* w_2 of an oriented vector bundle ξ over a paracompact base X as the pullback of the nonzero element under any classifying map $X\to BSO_n$. Any assignment of a cohomology class in $H^2(X,\mathbb{Z}_2)$ to oriented vector bundles ξ over paracompact spaces X satisfying

(i) naturality and

(ii) it gives the nonzero element in $H^2(BSO_n,\mathbb{Z}_2)$ if ξ is γ_n^+ (the canonical oriented rank n vector bundle over $G_n^+(\mathbb{R}^\infty)$)

must coincide with this one.

4. Open coverings and principal bundles

Let $P\to X$ be a principal G bundle, and consider a (say countable) open cover $\mathcal{U}$ of X, so that restricted to each of its open sets the bundle is trivial.

The transition maps between two of these trivializations define (say smooth) maps $g_{ij} : U_i \cap U_j \to G$, satisfying a 'cocycle condition':

$$g_{ij} g_{jk} = g_{ik} \text{ in } U_i \cap U_j \cap U_k.$$

Any open cover of X and an associated 1-cocycle define a G-principal bundle in a natural way.

Two 1-cocycles g, g' are *equivalent* (and define equivalent principal bundles) if there exists a '0-cocycle' for $\mathcal{U}$, $f_i : U_i \to G$, so that:

$$g'_{ij} = f_i^{-1} g_{ij} f_j \text{ in } U_i \cap U_j.$$

So we have the *cohomology set* $H^1(\mathcal{U}, G)$ of equivalence classes of G-valued 1-cocycles. A refinement $\mathcal{B}$ of $\mathcal{U}$ induces a natural map:

$$t : H^1(\mathcal{U}, G) \to H^1(\mathcal{B}, G).$$

The direct limit of the $H^1(\mathcal{U}, G)$ for varying covers, with respect to the maps t, is defined to be the 1-cohomology set $H^1(X, G)$.

Theorem (See [Husemoller] or [Hirzebruch]). The isomorphism classes of principal G-bundles over X are in 1-1 correspondence with the 1-cohomology set $H^1(X, G)$.

Exact sequences. The short exact sequences of groups:

$$SO_n \to O_n \to \mathbb{Z}_2,$$

where the first map is inclusion (injective) and the second map is quotient projection (surjective) induces in a natural way an exact sequence of 1-cohomology sets:

$$H^1(X; SO_n) \to H^1(X; O_n) \to H^1(X; \mathbb{Z}_2);$$

(note the last set is in fact the (Cech) 1-cohomology group.) Denote by ρ_* the last map. "Exact" here means the elements of $H^1(X; O_n)$ mapped to zero by ρ_* are exactly the subset $H^1(X; SO_n)$

This leads to a second definition of the first Stiefel-Whitney class of a rank n vector bundle E over X (assumed endowed with a metric on fibers). Consider the associated principal O_n bundle $P_O(E)$, of orthonormal frames of local sections of E. We have the class $[P_O(E)] \in H^1(X, O_n)$. Then let:

$$w_1(E) = \rho_*([P_O E]).$$

With this definition of w_1, clearly it vanishes exactly if the structure group of $P_O(E)$ can be reduced to SO_n, that is, exactly if E is orientable.

This definition clearly has the properties (i) naturality, with respect to bundle maps; (ii) when $E = \gamma^n$, the canonical rank n bundle over $BO_n = G_n(\mathbb{R}^\infty)$,it gives the nonzero element in $H^1(BO_n, \mathbb{Z}_2)$; otherwise γ^n would be orientable. By the classification theorem for vector bundles, this would imply *every* n-plane bundle is orientable, and we know that's false. As remarked earlier, this implies this definition of w_1 coincides with the original one.

5. Spin structures and the second Stiefel-Whitney class.

Definition. A spin structure on a principal SO_n bundle $P \to X$ is a pair (Q, Λ), where $Q \to X$ is a principal $Spin_n$ bundle and the bundle map $\Lambda : Q \to P$ is a two-fold covering map, commuting with the right actions of SO_n and $Spin_n$:

$$\Lambda(qg) = \Lambda(q)\lambda(g), \quad q \in Q, g \in Spin_n.$$

Here $\lambda : Spin_n \to SO_n$ is the standard two-fold covering map.

Homotopy criterion. Denote by α_F the image in $\pi_1(P)$ of the nonzero element in the fundamental group of a typical fiber $F \sim SO_n$ of P. Given a spin stucture (Q, Λ), denote by H the subgroup of index 2 of $\pi_1(P)$, image of $\pi_1(Q)$ under the map induced by Λ in π_1.

Proposition. $\alpha_F \notin H$.

In fact the existence of a subgroup of index 2 in $\pi_1(P)$ which does not contain α_F is necessary and sufficient for the existence of a spin structure on P (see [Friedrich].)

A subgroup $H \subset \pi_1(P)$ of index 2 defines a nontrivial homomorphism $f_H : \pi_1(P) \to \mathbb{Z}_2 \sim \pi_1(F)$. Considering the homotopy exact sequence:

$$\pi_2(X) \to \pi_1(F) \to \pi_1(P) \to \pi_1(X),$$

where $i_\# : \pi_1(F) \to \pi_1(P)$ is induced by inclusion, we see that $\alpha_F \notin H$ is equivalent to $f_H \circ i_\#$ being the identity on $\pi_1(F) \sim \mathbb{Z}_2$: the exact sequence splits. We conclude P admits a $Spin_n$ structure if, and only if:

$$\pi_1(P) \sim \pi_1(F) \oplus \pi_1(X) \sim \mathbb{Z}_2 \oplus \pi_1(X).$$

Cohomological criterion. An SO_n principal bundle $P \to X$ admits a $Spin_n$ structure iff there exists $f \in H^1(P, \mathbb{Z}_2)$ whose restriction to the cohomology of the fiber, $i^* f \in H^1(F; \mathbb{Z}_2) = \mathbb{Z}_2$ is nonzero.

This follows since an element of $H^1(P, \mathbb{Z}_2)$ is the same as a homomorphism $f : \pi_1(P) \to \mathbb{Z}_2$, and the condition $f \circ i_\# = Id$ is equivalent to $i^* f \neq 0$.

Note also that $H^1(P; \mathbb{Z}_2)$ is in bijective correspondence with 2-fold coverings of P.

Vector bundles and w_2. Def. A spin structure on an oriented rank n vector bundle $E \to X$ is a principal $Spin_n$ bundle $P_{Spin}(E)$ over X, together with a 2-fold covering map $\xi : P_{Spin}(E) \to P_{SO}(E)$, commuting with the right actions of $Spin_n$ and SO_n.

Thus the spin structures on E are in 1-1 correspondence with elements of $H^1(P_{SO}(E); \mathbb{Z}_2)$ with nonzero restriction to the fibers of $P_{SO}(E)$.

Connection with Stiefel-Whitney classes.

The short exact sequence of groups:

$$\mathbb{Z}_2 \to Spin_n \to SO_n$$

induces an exact sequence of cohomology sets:

$$H^1(X;\mathbb{Z}_2) \to H^1(X;Spin_n) \to H^1(X;SO_n) \to H^2(X;\mathbb{Z}_2).$$

The first two maps, induced by inclusion and quotient projection (resp.) are clear. The last one, from H^1 to H^2, is a 'coboundary map' δ, and may be described as follows (cp. [L-M, App. B].)

Let $(\mathcal{U}, g_{ij})$ be a covering of X and a $SO(n)$-valued cocycle $g \in Z^1(\mathcal{U}, SO_n)$ representing the SO_n bundle $P \to X$; assume each $U_i \cap U_j$ is simply-connected. Lift each g_{ij} to $\bar{g}_{ij} : U_i \cap U_j \to Spin_n$, and let:

$$w_{ijk} = \bar{g}_{ij}\bar{g}_{jk}\bar{g}_{ik}^{-1} : U_i \cap U_j \cap U_k \to Spin_n.$$

Then $w = w_{ijk}$ defines a $Spin_n$-valued 2-cocycle, with respect to the covering $\mathcal{U}$ of X, which projects to the cocycle $g_{ij}g_{jk}g_{ik}^{-1} \equiv Id$ in $H^1(\mathcal{U}, SO_n)$ (i.e. to the constant identity cocycle), and therefore in fact takes values in $\mathbb{Z}_2 = \pm 1 \subset Spin_n$. We take the cohomology class of $w \in Z^2(\mathcal{U}, \mathbb{Z}_2)$ to be the image under δ of the class $[g] \in H^1(\mathcal{U}; SO_n)$.

In particular, $w_{ijk} \sim 0$ means there exists a cocycle $h_{ij} : U_i \cap U_j \to \mathbb{Z}_2$ so that $w_{ijk} = h_{ij}h_{jk}h_{ik}^{-1}$ in $U_i \cap U_j \cap U_k$. It is easy to see this implies the $g'_{ij} = \bar{g}_{ij}h_{ij}^{-1}$ satisfy the cocycle condition, and hence define a $Spin_n$ principal bundle $Q = [g'] \in H^1(\mathcal{U}, Spin_n)$, which projects to the original $[g] \in H^1(\mathcal{U}; SO_n)$ defining P; i.e. P admits a $Spin_n$ structure.

In fact we claim that, for a rank n oriented vector bundle $E \to X$, the image under δ of the class in $H^1(X, SO_n)$ representing $P_{SO}(E)$ is the second Stiefel-Whitney class:

$$w_2(E) = \delta([P_{SO}(E)]).$$

As remarked earlier, we need to verify two conditions. First, that this definition is natural with respect to vector bundle morphisms; this is clear. Second, that it gives the nonzero element in $H^2(BSO_n, \mathbb{Z}_2) \sim \mathbb{Z}_2$, when E is the canonical oriented rank n bundle γ_+^n over BSO_n. But suppose we had $\delta([P_{SO}(\gamma_+^n)]) = 0$. This would mean this principal SO_n bundle over BSO_n (which is none other than the total space ESO_n of this universal bundle) would admit a Spin structure. As seen earlier, this would imply, for the fundamental groups: $\pi_1(ESO_n) = \pi_1(BSO_n) \oplus \mathbb{Z}_2$. This is impossible, since ESO_n is *contractible.*

Thus this definition of $w_2(E)$ coincides with the previously given one. We conclude:

An oriented vector bundle E (over a paracompact space X) admits a Spin structure $\Leftrightarrow w_2(E) = 0$.

REFERENCES:

T. Friedrich, *Dirac Operators in Riemannian Geometry*, AMS
F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag
D.Husemoller, *Fibre Bundles*, Springer-Verlag
H.B Lawson, M L. Michelson, *Spin Geometry*, Princeton U.P.
J. Milnor, J. Stasheff, *Characteristic Classes*, Princeton U.P.