GAKUTO International Series, Mathematical Sciences and Applications, Vol. 26 (2006)
Nonlinear Dispersive Equations, pp.189-200

Instability of standing waves for nonlinear Klein-Gordon equation and related system

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1 Introduction and Main Results

We study the strong instability of standing wave solutions $e^{i\omega t}\varphi(x)$ for the nonlinear Klein-Gordon equation of the form

$$\partial_t^2 u - \Delta u + u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1}$$

where $N \geq 2$, $1 < p < 1 + 4/(N-2)$, $-1 < \omega < 1$, and $\varphi \in H^1(\mathbb{R}^N)$ is a nontrivial solution of

$$-\Delta\varphi + (1-\omega^2)\varphi - |\varphi|^{p-1}\varphi = 0, \quad x \in \mathbb{R}^N. \tag{2}$$

From the result of Ginibre and Velo ([9]) the Cauchy problem for (1) is locally well-posed in the energy space $X := H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. Thus for any $(u_0, u_1) \in X$ there exists a unique solution $\vec{u} := (u, \partial_t u) \in C([0, T_{\max}); X)$ of (1) with $\vec{u}(0) = (u_0, u_1)$ such that either $T_{\max} = \infty$ (global existence) or $T_{\max} < \infty$ and $\lim_{t \to T_{\max}} \|\vec{u}(t)\|_X = \infty$ (finite time blowup). Moreover, the solution $u(t)$ satisfies the conservation laws of energy and charge:

$$E(\vec{u}(t)) = E(u_0, u_1), \quad Q(\vec{u}(t)) = Q(u_0, u_1), \quad t \in [0, T_{\max}),$$

[1]Research of the first author supported in part by JSPS grant 14740099
[2]Research of the second author supported in part by NSF grant DMS-0245578

where

$$E(u,v) = \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{3}$$

$$Q(u,v) = \operatorname{Im}\int_{\mathbb{R}^N} \overline{u}v\,dx. \tag{4}$$

Let $\phi_\omega \in H^1(\mathbb{R}^N)$ be the ground state (the least energy solution) of (2). We refer to [2, 30] for the existence of ϕ_ω, and to [12] for the uniqueness of ϕ_ω. The stability of standing waves $e^{i\omega t}\phi_\omega$ for (1) has been studied by many authors. First, we consider the orbital stability of $e^{i\omega t}\phi_\omega$. Shatah [27] proves that $e^{i\omega t}\phi_\omega$ is orbitally stable if $p < 1+4/N$ and $\omega_c < |\omega| < 1$, where

$$\omega_c = \sqrt{\frac{p-1}{4-(N-1)(p-1)}}. \tag{5}$$

Shatah and Strauss [29] prove that $e^{i\omega t}\phi_\omega$ is orbitally unstable when $p < 1+4/N$ and $|\omega| < \omega_c$ or when $p \geq 1+4/N$ and $|\omega| < 1$. Here, we say that a standing wave solution $e^{i\omega t}\varphi$ is orbitally stable for (1) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(u_0, u_1) \in X$ satisfies $\|(u_0,u_1) - (\varphi, i\omega\varphi)\|_X < \delta$, then the solution $u(t)$ of (1) with $\vec{u}(0) = (u_0, u_1)$ exists globally and satisfies

$$\sup_{t\geq 0} \inf_{\theta\in\mathbb{R}, y\in\mathbb{R}^N} \|\vec{u}(t) - e^{i\theta}(\varphi(\cdot+y), i\omega\varphi(\cdot+y))\|_X < \varepsilon.$$

Otherwise, $e^{i\omega t}\varphi$ is said to be orbitally unstable.

Next, we consider instability of $u_\omega(t)$ in stronger sense. Berestycki and Cazenave [1] prove that the ground state standing wave $e^{i\omega t}\phi_\omega$ for the nonlinear Klein-Gordon equations (1) is very strongly unstable (see Definition 1 below) when the frequency $\omega = 0$ (see also [26]). Shatah [28] proves that the ground state standing wave $e^{i\omega t}\phi_\omega$ for the nonlinear Klein-Gordon equations with general nonlinearity is strongly unstable (see Definition 2 below) when $\omega = 0$ and $N \geq 3$. Recently, the authors in [22] prove that the ground state standing waves $e^{i\omega t}\phi_\omega$ for the nonlinear Klein-Gordon equation (1) are very strongly unstable when the frequency $|\omega| \leq \sqrt{(p-1)/(p+3)}$ and $N \geq 3$. Here, we give the definitions of very strong instability and strong instability.

Definition 1 (very strong instability) We say that $e^{i\omega t}\varphi$ is *very strongly unstable* for (1) if for any $\varepsilon > 0$ there exists $(u_0, u_1) \in X$ such that $\|(u_0,u_1) - (\varphi, i\omega\varphi)\|_X < \varepsilon$ and the solution $u(t)$ of (1) with $\vec{u}(0) = (u_0, u_1)$ blows up in finite time.

Definition 2 (strong instability) We say that $e^{i\omega t}\varphi$ is *strongly unstable* for (1) if for any $\varepsilon > 0$ there exists $(u_0, u_1) \in X$ such that $\|(u_0,u_1) - (\varphi, i\omega\varphi)\|_X < \varepsilon$ and the solution $u(t)$ of (1) with $\vec{u}(0) = (u_0, u_1)$ either blows up in finite time or exists globally and satisfies $\limsup_{t\to\infty} \|\vec{u}(t)\|_X = \infty$.

Note that, by the definitions, if $e^{i\omega t}\varphi$ is very strongly unstable then it is strongly unstable, and that if $e^{i\omega t}\varphi$ is strongly unstable then it is orbitally unstable.

Before stating our main results, we recall instability results for the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{6}$$

Let $\omega > 0$ and $\phi_\omega \in H^1(\mathbb{R}^N)$ be the ground state of

$$-\Delta\varphi + \omega\varphi - |\varphi|^{p-1}\varphi = 0, \quad x \in \mathbb{R}^N. \tag{7}$$

It is known that for any $\omega > 0$ the standing wave solution $e^{i\omega t}\phi_\omega$ for (6) is orbitally stable when $1 < p < 1+4/N$, and it is very strongly unstable when $1+4/N < p < 1+4/(N-2)$ (see [1, 7]). Moreover, for the critical case $p = 1+4/N$, for any $\omega > 0$ and any nontrivial solution $\varphi \in H^1(\mathbb{R}^N)$ of (7), it is known that the standing wave $e^{i\omega t}\varphi$ is very strongly unstable for (6) (see [32]). For general theory of orbital stability and instability of solitary waves, we refer to Grillakis, Shatah and Strauss [10, 11].

We state our main results.

Theorem 1 *Let $N \geq 2$, $1 < p < 1+4/(N-2)$, $\omega \in (-1,1)$ and ϕ_ω be the ground state of* (2)*. Assume that $|\omega| \leq \omega_c$ if $p < 1+4/N$, where the critical frequency ω_c is given by* (5)*. Then, the standing wave $e^{i\omega t}\phi_\omega$ for nonlinear Klein-Gordon equation* (1) *is strongly unstable in the sense of Definition 2.*

Can we refine further this instability result? Namely, can we prove in certain cases that standing wave $e^{i\omega t}\phi_\omega$ for (1) is very strongly unstable in the sense of Definition 1? The result of Cazenave [5] gives an answer of this question for the restricted range for the exponent p of nonlinearity $1 < p \leq 5$ for $N = 2$ and $1 < p \leq N/(N-2)$ for $N \geq 3$. Cazenave proves that any global solution $u(t)$ of (1) is uniformly bounded in X, i.e., $\sup_{t\geq 0} \|\vec{u}(t)\|_X < \infty$, if $1 < p \leq 5$ and $N = 2$, and if $1 < p \leq N/(N-2)$ and $N \geq 3$. Therefore, for this range of the exponent p, Theorem 1 together with the result of Cazenave gives us a very strongly instability result in the sense of Definition 1 for ground state standing waves $e^{i\omega t}\phi_\omega$ of the NLKG equation (1). Using an argument in Merle and Zaag [17], we can extend the result of Cazenave and prove the uniform boundedness of global solutions of (1) in X when $1 < p < 1+4/(N-1)$ and $N \geq 2$. The following Lemma holds.

Lemma 2 *Let $N \geq 2$ and $1 < p < 1+4/(N-1)$. If $\vec{u} \in C([0,\infty), X)$ is a global solution of* (1)*, then $\sup_{t\geq 0} \|\vec{u}(t)\|_X < \infty$.*

Therefore, from Theorem 1 and Lemma 2 we deduce the following.

Corollary 3 *In addition to the assumptions in Theorem 1, let* $1 < p \le 1 + 4/(N-1)$ *if* $N = 2, 3$, *and that* $1 < p < 1 + 4/(N-1)$ *if* $N \ge 4$. *Then, the ground state standing wave* $e^{i\omega t}\phi_\omega$ *for the nonlinear KG equation* (1) *is very strongly unstable in the sense of Definition 1.*

Remark. Let us mention that when the exponent p of nonlinearity is in the range $1 + 4/(N-1) < p < 1 + 4/(N-2)$ we can not give better instability results than those in Theorem 1 for ground state standing waves $e^{i\omega t}\phi_\omega$ of the nonlinear KG equation (1).

For the critical frequency $\omega = \omega_c$ in the case $1 < p < 1 + 4/N$, we have the following.

Theorem 4 *Let* $N \ge 2$, $1 < p < 1 + 4/N$ *and* $\varphi \in H^1(\mathbb{R}^N)$ *be any nontrivial, radially symmetric solution of* (2) *with* $\omega = \omega_c$. *Then, the standing wave solution* $e^{i\omega_c t}\varphi$ *of nonlinear KG equation* (1) *is very strongly unstable in the sense of Definition 1. The same assertion is true for* $\omega = -\omega_c$.

For the existence of infinitely many radially symmetric solutions of (2), we refer to [3].

As mentioned above, a similar result of Theorem 4 is known for the nonlinear Schrödinger equation (6) (see [32]) in the critical case $p = 1 + 4/N$, without assuming the radial symmetry of solution of (7) and the restriction on space dimensions $N \ge 2$.

The proofs of Theorems 1 and 4 are based on using local versions of the virial type identities.

To prove instability of the ground state Shatah in [28] considers a local version of the following identity

$$\frac{d}{dt}\operatorname{Re}\int_{\mathbb{R}^N} x\cdot\nabla u\,\partial_t\bar{u}\,dx = NK_1(\vec{u}(t)), \tag{8}$$
$$K_1(u,v) := -\frac{1}{2}\|v\|_2^2 + \left(\frac{1}{2} - \frac{1}{N}\right)\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$

Since the integral in the left-hand side of (8) is not well-defined on the energy space X, we need to approximate the weight function x in (8) by suitable bounded functions. To control error terms by the approximation, initial perturbations are restricted to being radially symmetric and the decay estimate for radially symmetric functions in $H^1(\mathbb{R}^N)$:

$$\|w\|_{L^\infty(|x|\ge m)} \le Cm^{-(N-1)/2}\|w\|_{H^1} \tag{9}$$

(see [30]) is employed. The assumption $N \ge 2$ is needed here. This kind of approach has been also used for blowup problem of NLS (6) (see, e.g., [20, 21, 14, 15, 16, 18, 19]).

In the proof of Theorem 1 for the case $p \ge 1 + 4/N$, we use a local version of the virial identity

$$-\frac{d}{dt}\operatorname{Re}\int_{\mathbb{R}^N}\{2x\cdot\nabla u + Nu\}\partial_t\bar{u}\,dx = P(u(t)), \tag{10}$$

where

$$P(u) := 2\|\nabla u\|_2^2 - \frac{N(p-1)}{p+1}\|u\|_{p+1}^{p+1}.$$

Note that (10) follows from (8) and

$$\text{(11)} \qquad \begin{aligned} &\frac{1}{2}\frac{d^2}{dt^2}\|u(t)\|_2^2 = \frac{d}{dt}\operatorname{Re}\int_{\mathbb{R}^N} u\partial_t\overline{u}\,dx = -K_2(\vec{u}(t)), \\ &K_2(u,v) = -\|v\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2 - \|u\|_{p+1}^{p+1}, \end{aligned}$$

and that the functional P appears in the virial identity for the nonlinear Schrödinger equation (6):

$$\text{(12)} \qquad \frac{d^2}{dt^2}\|xu(t)\|_2^2 = 4P(u(t)).$$

For the case $p < 1 + 4/N$, we use a local version of the identity

$$\text{(13)} \qquad -\frac{d}{dt}\operatorname{Re}\int_{\mathbb{R}^N}\{2x\cdot\nabla u + (N+\alpha)u\}\partial_t\overline{u}\,dx = K(\vec{u}(t)),$$

where $\alpha := 4/(p-1) - N$ and

$$\text{(14)} \qquad K(u,v) := -\alpha\|v\|_2^2 + \alpha\|u\|_2^2 + (\alpha+2)\{\|\nabla u\|_2^2 - \frac{2}{p+1}\|u\|_{p+1}^{p+1}\}$$

(cf. [29, page 185]). Note that

$$\text{(15)} \qquad \begin{aligned} K(u,v) &= P(u) + \alpha K_2(u,v) \\ &= -2(\alpha+1)\|v - i\omega u\|_2^2 + 2(\alpha+2)(E-\omega Q)(u,v) \\ &\qquad -2\alpha\omega Q(u,v) - 2\{1-(\alpha+1)\omega^2\}\|u\|_2^2, \end{aligned}$$

and that $1-(\alpha+1)\omega^2 \geq 0$ if and only if $|\omega| \leq \omega_c$.

Further, we consider the Klein-Gordon-Zakharov system

$$\text{(16)} \qquad \partial_t^2 u - \Delta u + u + nu = 0, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^N,$$

$$\text{(17)} \qquad c_0^{-2}\partial_t^2 n - \Delta n = \Delta(|u|^2), \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^N,$$

where $N = 2, 3$ and $c_0 > 0$ is a constant. The system (16)-(17) describes the interaction of Langumiur waves and ion acoustic waves in a plasma. The complex valued function u denotes the fast time scale component of electric field raised by electrons, and the real valued function n denotes the deviation of ion density (see [34, 4, 8]).

We consider instability of standing wave solutions

$$(u_\omega(t,x), n_\omega(t,x)) = (e^{i\omega t}\phi_\omega(x), -|\phi_\omega(x)|^2)$$

for (16)-(17), where $-1 < \omega < 1$, and $\phi_\omega \in H^1(\mathbb{R}^N)$ is the ground state of

$$-\Delta\varphi + (1-\omega^2)\varphi - |\varphi|^2\varphi = 0, \quad x \in \mathbb{R}^N. \tag{18}$$

The well-posedness of the Cauchy problem for (16)-(17) in the energy space is studied by Ozawa, Tsutaya and Tsutsumi [25]. Here, the energy space Y is defined by $Y = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times \dot{H}^{-1}(\mathbb{R}^N)$. When $N = 3$ and $c_0 \neq 1$, it is proved in [25] that for any $(u_0, u_1, n_0, n_1) \in Y$ there exists a unique solution $\mathbf{u} := (u, \partial_t u, n, \partial_t n) \in C([0, T_{\max}); Y)$ of (16)-(17) with initial data $\mathbf{u}(0) = (u_0, u_1, n_0, n_1)$ satisfying the conservation laws of the energy $H(\mathbf{u}(t)) = H(\mathbf{u}(0))$ and the charge $Q(\mathbf{u}(t)) = Q(\mathbf{u}(0))$ for all $t \in [0, T_{\max})$, where Q is defined by (4), and

$$\begin{aligned} H(u, v, n, \nu) = {} & \frac{1}{2}\|v\|_2^2 + \frac{1}{4c_0^2}\|\nu\|_{\dot{H}^{-1}}^2 \\ & + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{1}{4}\|n\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 n\, dx. \end{aligned} \tag{19}$$

For global existence results for the case $c_0 = 1$, see [24] and [31].

By a similar method as in the proof of Theorem 1 for the case $p \geq 1 + 4/N$ together with an argument in Merle [16] for the Zakharov system, we have the following.

Theorem 5 *Let $N = 2, 3$, $\omega \in (-1, 1)$ and ϕ_ω be the ground state of (18). Then, the standing wave $(e^{i\omega t}\phi_\omega, -|\phi_\omega|^2)$ of KGZ system (16)-(17) is strongly unstable in the following sense. For any $\lambda > 1$, the solution $\mathbf{u}(t)$ of (16)-(17) with initial data $\mathbf{u}(0) = (\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)$ either blows up in finite time or exists globally and satisfies $\limsup_{t\to\infty} \|\mathbf{u}(t)\|_Y = \infty$.*

Remark. It is known (see [4]) that the negative initial energy $H(\mathbf{u}(0))$ implies that the solution $\mathbf{u}(t)$ of (16)-(17) either blows up in finite time or blows up in infinite time, namely the solution exists globally and satisfies the asymptotic condition $\limsup_{t\to\infty} \|\mathbf{u}(t)\|_Y = \infty$. Since the energy

$$H(\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0) > 0$$

for λ close to 1, the result in [4] is not applicable to Theorem 5.

Next, we consider the very strong instability of $(e^{i\omega t}\phi_\omega, -|\phi_\omega|^2)$ for the system (16)-(17). Since the second equation (17) of the Klein-Gordon-Zakharov system is massless, it seems difficult to obtain an uniform boundedness of global solutions for (16)-(17) similar to Lemma 2. Therefore, for the standing wave $(e^{i\omega t}\phi_\omega, -|\phi_\omega|^2)$ we do not deduce a very strong instability similar to the instability result in corollary 3 of Theorem 1. However, using the method in our previous paper [22], we obtain the following very strong instability result for small frequencies.

Theorem 6 *Let $N = 3$, $|\omega| < 1/\sqrt{3}$ and ϕ_ω be the ground state of* (18)*. Then, the standing wave $(e^{i\omega t}\phi_\omega, -|\phi_\omega|^2)$ of the KGZ system* (16)*-*(17) *is very strongly unstable in the following sense. For any $\lambda > 1$, the solution $\mathbf{u}(t)$ of* (16)*-*(17) *with the initial data $\mathbf{u}(0) = (\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)$ blows up in a finite time.*

In the next section, we give a sketch of the proof of Theorem 1. For the proofs of Lemma 2, Theorems 4, 5 and 6, see [23].

2 Outline of the proof of Theorem 1

Here, we present a short sketch of the proof of Theorem 1. For the details, see [23].

First, we consider the case $p \geq 1 + 4/N$. We define

$$J_\omega(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1-\omega^2}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{20}$$

$$d_\omega^1 = \inf\{J_\omega(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\},\ P(u) = 0\}, \tag{21}$$

$$\mathcal{R}_\omega^1 = \{(u,v) \in X : (E - \omega Q)(u,v) < d_\omega^1,\ P(u) < 0\}. \tag{22}$$

Note that

$$(E - \omega Q)(u,v) = J_\omega(u) + \frac{1}{2}\|v - i\omega u\|_2^2, \tag{23}$$

$$P(u) = 2\partial_\lambda J_\omega(\lambda^{N/2}u(\lambda\cdot))|_{\lambda=1}. \tag{24}$$

The following Lemmas are crucial for the proof of Theorem 1 in this case.

Lemma 7 *Let $N \geq 2$, $1 + 4/N \leq p < 1 + 4/(N-2)$ and $\omega \in (-1,1)$. Then, we have the followings.*

(i) $J_\omega(u) - \dfrac{1}{N(p-1)}P(u) > d_\omega^1$ *for all $u \in H^1(\mathbb{R}^N)$ satisfying $P(u) < 0$.*

(ii) *The minimization problem* (21) *is attained at the ground state ϕ_ω of* (2)*.*

(iii) $\lambda(\phi_\omega, i\omega\phi_\omega) \in \mathcal{R}_\omega^1$ *for all $\lambda > 1$.*

Lemma 8 *Suppose that $N \geq 2$, $1 + 4/N \leq p < 1 + 4/(N-2)$ and $\omega \in (-1,1)$. If $(u_0, u_1) \in \mathcal{R}_\omega^1$, then the solution $u(t)$ of* (1) *with $\vec{u}(0) = (u_0, u_1)$ satisfies*

$$-\frac{1}{N(p-1)}P(u(t)) > d_\omega^1 - (E - \omega Q)(u_0, u_1), \quad t \in [0, T_{\max}). \tag{25}$$

Proof of Theorem 1 for the case $p \geq 1 + 4/N$**.** Let $\lambda > 1$ and put

$$\delta := \frac{N(p-1)}{2}\{d_\omega^1 - (E - \omega Q)(\lambda(\phi_\omega, i\omega\phi_\omega))\}.$$

Then, by Lemma 7 (iii), we have $\delta > 0$. Suppose that the solution $u(t)$ of (1) with $\vec{u}(0) = \lambda(\phi_\omega, i\omega\phi_\omega)$ exists for all $t \in [0, \infty)$ and is uniformly bounded in X, i.e.,

$$M_1 := \sup_{t \geq 0} \|\vec{u}(t)\|_X < \infty. \tag{26}$$

Note that $u(t)$ is radially symmetric in x for all $t \geq 0$. For the solution $u(t)$ of (1) and $m > 0$, we define a function $I_m^1(t)$ by

$$I_m^1(t) = 2\,\mathrm{Re}\int_{\mathbb{R}^N} \Psi_m \partial_r u \partial_t \overline{u}\, dx + \mathrm{Re}\int_{\mathbb{R}^N} \Phi_m u \partial_t \overline{u}\, dx, \tag{27}$$

where

$$\Phi_m(r) = \Phi\left(\frac{r}{m}\right), \quad \Psi_m(r) = \frac{1}{r^{N-1}}\int_0^r s^{N-1}\Phi_m(s)\, ds \tag{28}$$

and $\Phi \in C^2([0, \infty))$ is a non-negative function such that

$$\Phi(r) = \begin{cases} N & \text{for } \ 0 \leq r \leq 1, \\ 0 & \text{for } \ r \geq 2, \end{cases} \qquad \Phi'(r) \leq 0 \ \text{ for } \ 1 \leq r \leq 2.$$

Then, we have

$$\begin{aligned} &-\frac{d}{dt} I_m^1(t) \\ &= 2\int_{\mathbb{R}^N} \Psi_m' |\nabla u|^2\, dx - \frac{p-1}{p+1}\int_{\mathbb{R}^N} \Phi_m |u|^{p+1}\, dx - \frac{1}{2}\int_{\mathbb{R}^N} \Delta\Phi_m |u|^2\, dx \\ &\leq P(u(t)) + \frac{N(p-1)}{p+1}\int_{|x| \geq m} |u(t,x)|^{p+1}\, dx + \frac{C_0}{m^2}\|u(t)\|_2^2 \end{aligned}$$

for all $t \geq 0$, where C_0 is a positive constant independent of m. By (9) and (26), we have

$$\begin{aligned} &\int_{|x| \geq m} |u(t,x)|^{p+1}\, dx \leq \|u(t)\|_{L^\infty(|x| \geq m)}^{p-1} \|u(t)\|_2^2 \\ &\leq C m^{-(N-1)(p-1)/2} \|u(t)\|_{H^1}^{p+1} \leq C M_1^{p+1} m^{-(N-1)(p-1)/2} \end{aligned}$$

for all $t \geq 0$ and $m > 0$. Note that we assume $N \geq 2$. Thus, there exists $m_0 > 0$ such that

$$\sup_{t \geq 0}\left(\frac{N(p-1)}{p+1}\int_{|x| \geq m_0} |u(t,x)|^{p+1}\, dx + \frac{C_0}{m_0^2}\|u(t)\|_2^2\right) < \delta.$$

Then, by Lemma 8, we have $(d/dt)I_{m_0}^1(t) \geq \delta$ for all $t \geq 0$, which implies $\lim_{t\to\infty} I_{m_0}^1(t) = \infty$. On the other hand, there exists a constant $C = C(m_0) > 0$ such that $I_{m_0}^1(t) \leq$

$C\|\vec{u}(t)\|_X^2 \le CM_1^2$ for all $t \ge 0$. This is a contradition. Hence, for any $\lambda > 1$, the solution $u(t)$ of (1) with $\vec{u}(0) = \lambda(\phi_\omega, i\omega\phi_\omega)$ either blows up in finite time or exists for all $t \ge 0$ and $\limsup_{t\to\infty} \|\vec{u}(t)\|_X = \infty$. This completes the proof of Theorem 1 for the case $p \ge 1 + 4/N$. □

Next, we consider the case where $p < 1 + 4/N$. For this case, we need a variational characterization of the ground state ϕ_ω of (2) different from that of the case $p \ge 1+4/N$. We consider

$$K_\omega^0(u) = \alpha(1-\omega^2)\|u\|_2^2 + (\alpha+2)\{\|\nabla u\|_2^2 - \frac{2}{p+1}\|u\|_{p+1}^{p+1}\}, \tag{29}$$

$$d_\omega^0 = \inf\{J_\omega(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\},\ K_\omega^0(u) = 0\}, \tag{30}$$

$$\mathcal{R}_\omega^0 = \{(u,v) \in X : (E - \omega Q)(u,v) < d_\omega^0,\ K_\omega^0(u) < 0\}, \tag{31}$$

where $\alpha = 4/(p-1) - N > 0$. Note that

$$K_\omega^0(u) = 2\partial_\lambda J_\omega(\lambda^\beta u(\lambda \cdot))|_{\lambda=1}, \quad \beta = \frac{\alpha+N}{2} = \frac{2}{p-1}.$$

The following Lemmas play an essential role in the proof of Theorem 1 for the case $p < 1 + 4/N$, as Lemmas 7 and 8 do for the case $p \ge 1 + 4/N$.

Lemma 9 *Let $N \ge 2$, $1 < p < 1 + 4/N$ and $\omega \in (-1,1)$. Then, we have the following:*

(i) $\dfrac{1-\omega^2}{\alpha+2}\|u\|_2^2 > d_\omega^0$ *for all $u \in H^1(\mathbb{R}^N)$ satisfying $K_\omega^0(u) < 0$.*

(ii) *The minimization problem* (30) *is attained at the ground state ϕ_ω of* (2).

(iii) $\lambda(\phi_\omega, i\omega\phi_\omega) \in \mathcal{R}_\omega^0$ *for all $\lambda > 1$.*

Lemma 10 *Suppose that $N \ge 2$, $1 < p < 1 + 4/N$ and $\omega \in (-1,1)$. If $(u_0, u_1) \in \mathcal{R}_\omega^0$, then the solution $u(t)$ of* (1) *with $\vec{u}(0) = (u_0, u_1)$ satisfies*

$$\frac{1-\omega^2}{\alpha+2}\|u(t)\|_2^2 > d_\omega^0, \quad t \in [0, T_{\max}).$$

Now, instead of $I_m^1(t)$ defined by (27), we consider a function

$$I_m^2(t) = I_m^1(t) + \alpha \operatorname{Re} \int_{\mathbb{R}^N} u \partial_t \bar{u}\, dx, \tag{32}$$

where $\alpha := 4/(p-1) - N$. Using this function $I_m^2(t)$ and based on Lemmas 9 and 10, we can prove Theorem 1 for the case $p < 1 + 4/N$ in a way similar to the case $p \ge 1 + 4/N$.

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