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# STRONG INSTABILITY OF STANDING WAVES FOR NONLINEAR KLEIN-GORDON EQUATIONS

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**Abstract.** The strong instability of ground state standing wave solutions $e^{i\omega t}\phi_\omega(x)$ for nonlinear Klein-Gordon equations has been known only for the case $\omega = 0$. In this paper we prove the strong instability for small frequency $\omega$.

1. **Introduction and Results.** We consider the strong instability of the ground state standing wave solutions $e^{i\omega t}\phi_\omega(x)$ for nonlinear Klein-Gordon equation of the form

$$\partial_t^2 u - \Delta u + u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.1}$$

where $n \geq 3$, $1 < p < 1 + 4/(n-2)$, $\omega \in (-1,1)$, and $\phi_\omega(x)$ is the ground state, i.e., the unique positive radially symmetric solution in $H^1(\mathbb{R}^n)$ of

$$-\Delta\phi + (1-\omega^2)\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^n \tag{1.2}$$

(see Strauss [17] and Berestycki and Lions [2] for the existence, and Kwong [8] for the uniqueness of $\phi_\omega$). The stability and instability of the ground state standing waves $e^{i\omega t}\phi_\omega(x)$ for (1.1) have been studied by many authors. Berestycki and Cazenave [1] proved that $e^{i\omega t}\phi_\omega(x)$ are strongly unstable when $\omega = 0$ and $1 < p < 1 + 4/(n-2)$ (see also Payne and Sattinger [13] and Shatah [15]). Shatah (see [14]) proved that the ground state standing waves $e^{i\omega t}\phi_\omega(x)$ are orbitally stable when $1 < p < 1+4/n$ and $\omega_c < |\omega| < 1$, where the critical frequency $\omega_c$ is

$$\omega_c = \sqrt{\frac{p-1}{4-(n-1)(p-1)}}. \tag{1.3}$$

Shatah and Strauss [16] proved that $e^{i\omega t}\phi_\omega(x)$ are orbitally unstable when $1 < p < 1 + 4/n$ and $|\omega| < \omega_c$ or when $p \geq 1 + 4/n$ and $\omega \in (-1,1)$. For related results for nonlinear Schrödinger equations, see [1, 3, 4, 18, 19], and for general theory of orbital stability and instability of solitary waves, see Grillakis, Shatah and Strauss

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[6, 7] and [12]. Here, we give the definition of orbital stability/instability and blow-up instability of $e^{i\omega t}\phi_\omega(x)$. Orbital stability refers to stability up to translations and phase shifts. More precisely

**Definition of orbital stability** We say that the standing wave $e^{i\omega t}\phi_\omega(x)$ is orbitally stable for (1.1) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ satisfies

$$\inf_{\theta\in\mathbb{R}, y\in\mathbb{R}^n} \|(u_0(.), u_1(.)) - (e^{i\theta}\phi_\omega(. + y), i\omega e^{i\theta}\phi_\omega(. + y))\|_{H^1\times L^2} < \delta,$$

then the solution $u(t, x)$ of (1.1) with data $(u_0, u_1)$ exists globally in time and satisfies

$$\sup_{t\geq 0} \inf_{\theta\in\mathbb{R}, y\in\mathbb{R}^n} \|(u(t, .), \partial_t u(t, .)) - (e^{i\theta}\phi_\omega(. + y), i\omega e^{i\theta}\phi_\omega(. + y))\|_{H^1\times L^2} < \varepsilon.$$

Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be orbitally unstable.

**Definition of strong blow-up instability** We say that the standing wave $e^{i\omega t}\phi_\omega(x)$ is strongly blow-up unstable if for any $\varepsilon > 0$ there exists $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that

$$\|(u_0(.), u_1(.)) - (\phi_\omega(.), i\omega\phi_\omega(.))\|_{H^1\times L^2} < \varepsilon$$

and the solution $u(t, x)$ of (1.1) with data $(u_0, u_1)$ blows up in a finite time.

From the above definitions of instability, if the standing wave $e^{i\omega t}\phi_\omega(x)$ is strongly blow-up unstable then it is orbitally unstable as well. We note that the strong instability of ground state standing waves $e^{i\omega t}\phi_\omega(x)$ has not been known except for the case of frequency $\omega = 0$.

The main result in this paper is as follows.

**Theorem 1.** *Let $n \geq 3$, $1 < p < 1 + 4/(n-2)$, $\omega \in (-1, 1)$, and $\phi_\omega(x)$ be the unique positive radially symmetric solution of* (1.2). *If $|\omega| \leq \sqrt{(p-1)/(p+3)}$, then the standing wave solution $e^{i\omega t}\phi_\omega(x)$ for* (1.1) *is strongly blow-up unstable.*

**Remark 1** When $p \geq 1 + 4/n$ and the frequency $\omega$ is close to 1 the standing waves $e^{i\omega t}\phi_\omega(x)$ for NLKG (1.1) are again strongly blow-up unstable (forthcoming paper Ohta and Todorova).

**Remark 2** It is an interesting problem whether or not there is a frequency $\omega$ such that the standing wave $e^{i\omega t}\phi_\omega(x)$ is orbitally unstable for (1.1) but not strongly blow-up unstable.

The idea of the proof of Theorem 1 is the following. By the modulation $v(t, x) = e^{-i\omega t}u(t, x)$, the nonlinear Klein-Gordon equation (1.1) is transformed to the following perturbed Schrödinger equation

$$\partial_t^2 v + 2\omega i\partial_t v - \Delta v + (1 - \omega^2)v = |v|^{p-1}v, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n. \tag{1.4}$$

So, for $\gamma \in \mathbb{R}$ and $m > 0$, we consider

$$\partial_t^2 u + 2\gamma i\partial_t u - \Delta u + m^2 u = |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.5}$$

and the stationary problem

$$-\Delta\phi + m^2\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^n. \tag{1.6}$$

Then Theorem 1 follows from

**Theorem 2.** *Let $n \geq 3$, $1 < p < 1+4/(n-2)$, $\gamma \in \mathbb{R}$, $m > 0$ and $\psi(x)$ be the unique positive radially symmetric solution of* (1.6) *in $H^1(\mathbb{R}^n)$. If $4\gamma^2 \leq (p-1)m^2$, then the stationary solution $\psi(x)$ for* (1.5) *is strongly blow-up unstable in the following sense. For any $\lambda > 1$, the solution $u(t,x)$ of* (1.5) *with data $(\lambda\psi, 0)$ blows up in a finite time.*

In the next section, we give the proof of Theorem 2. For the special case $\gamma = 0$ the result of Theorem 2 was proved by Payne and Sattinger [13]. Their proof is based on the "potential well" arguments. The crucial point in their proof is the invariance under the flow of the set $\Sigma_1$ defined by (2.4). However, when the frequency $\gamma \neq 0$, new terms appear in (2.5) and (2.6), and we need to modify the argument in [13]. To control those terms, we need not only the invariant set $\Sigma_1$ but we also introduce another invariant set $\Sigma_2$ defined by (2.4) and consider the intersection $\Sigma_1 \cap \Sigma_2$ of these two invariant under the flow sets. The restriction for the space dimension $n \geq 3$ in Theorems 1 and 2 comes from the variational characterization (2.2) of $\psi(x)$ with $j = 2$. Finally, we note that the functional $K_2$ and the invariant set $\Sigma_2$ have been used by many authors (see, e.g., [14, 15, 16]), and the idea for using the intersection of appropriate two or more invariant sets was applied by Liu [10, 11] for the generalized Kadomtsev-Petviashvili equations.

## 2. **Proof of Theorem 2.**

For $(u,v) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, we define

$$E(u,v) = \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{m^2}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$
$$Q(u,v) = \operatorname{Im}\int_{\mathbb{R}^n} v(x)\overline{u(x)}\,dx + \gamma\|u\|_2^2.$$

The local existence and uniqueness of the Cauchy problem (1.5) in the energy space yields in the following way. Due to Ginibre and Velo [5] we have the local existence and uniqueness of solutions $u(t,x)$ in the energy space for the NLKG (1.1). Then the modulation $v(t,x) = e^{-i\omega t}u(t,x)$, implies the local existence and uniqueness in the energy space of the Cauchy problem (1.5). That is, for any data $(u_0,u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, there exist $T = T(\|(u_0,u_1)\|_{H^1 \times L^2}) > 0$ and a unique solution $u(t,x)$ of (1.5) with data $(u_0,u_1)$ such that

$$(u, \partial_t u) \in C([0,T], H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)).$$

Moreover, the energy $E(t)$ and the charge $Q(t)$ are conserved quantities of (1.5), namely

$$E(u(t), \partial_t u(t)) = E(u_0,u_1),\ Q(u(t), \partial_t u(t)) = Q(u_0,u_1) \quad (0 \leq t \leq T).$$

To obtain the conservation of energy we multiply the equation (1.5) by $\overline{\partial_t u}$, integrate over $\mathbb{R}^n$ and take the real part. To obtain the conservation of charge we multiply (1.5) by $\overline{u}$, integrate over $\mathbb{R}^n$ and take the imaginary part.

For $u \in H^1(\mathbb{R}^n)$, we define the functionals

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{m^2}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$
$$K_1(u) = \|\nabla u\|_2^2 + m^2\|u\|_2^2 - \|u\|_{p+1}^{p+1},$$
$$K_2(u) = \left(\frac{1}{2} - \frac{1}{n}\right)\|\nabla u\|_2^2 + \frac{m^2}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$
$$J_1(u) = J(u) - \frac{1}{p+1}K_1(u) = \frac{p-1}{2(p+1)}(\|\nabla u\|_2^2 + m^2\|u\|_2^2),$$
$$J_2(u) = J(u) - K_2(u) = \frac{1}{n}\|\nabla u\|_2^2.$$

By exact calculations one can observe that

$$K_1(u) = \partial_\lambda J(\lambda u)|_{\lambda=1}, \quad K_2(u) = \frac{1}{n}\partial_\lambda J(u(\cdot/\lambda))|_{\lambda=1}.$$

Let $\psi$ be the bound state of equation (1.6), i.e. the unique positive radially symmetric solution of (1.6).

**Lemma 3.** *Let $n \geq 3$, $1 < p < 1 + 4/(n-2)$, $m > 0$. Consider the minimization problems*

$$d_j = \inf\{J_j(u) : \; u \in H^1_{rad}(\mathbb{R}^n) \setminus \{0\}, \; K_j(u) = 0\}, \quad j = 1, 2 \tag{2.1}$$

*and*

$$\tilde{d}_j = \inf\{J_j(u) : \; u \in H^1_{rad}(\mathbb{R}^n) \setminus \{0\}, \; K_j(u) \leq 0\}, \quad j = 1, 2. \tag{2.2}$$

*Then*

$$d_j = \tilde{d}_j, \quad j = 1, 2 \tag{2.3}$$

*are attained at the unique positive radially symmetric solution $\psi(x)$ of* (1.6) *in $H^1(\mathbb{R}^n)$. Moreover, $d_1 = d_2 = J(\psi)$, $J'(\psi) = 0$ and $K_1(\psi) = K_2(\psi) = 0$.*

*Proof.* It is known that the identity (2.3) holds for $j = 2$, and the infimum of the minimization problems is attained at a positive function $\psi_2(x)$ in $H^1_{rad}(\mathbb{R}^n)$ which is a solution of (1.6) (see [15]).

Below, we prove Lemma 3 for the case $j = 1$. First, we show that the identity (2.3) holds for $j = 1$. By the definition of $d_1$ and $\tilde{d}_1$ we have $\tilde{d}_1 \leq d_1$. On the other hand, for any $v \in H^1_{rad}(\mathbb{R}^n)$ satisfying $K_1(v) < 0$, there exists $\lambda_0 \in (0, 1)$ such that $K_1(\lambda_0 v) = 0$, because $K_1(\lambda v) = K_1(v) < 0$ for $\lambda = 1$ and $K_1(\lambda v) > 0$ for $\lambda$ close to 0. Then, we have $d_1 \leq J_1(\lambda_0 v) = \lambda_0^2 J_1(v) < J_1(v)$, which implies $d_1 \leq \tilde{d}_1$. Thus, the identity $d_1 = \tilde{d}_1$ holds. Next, we show that the infimum of the minimization problem (2.2) for $j = 1$ is attained at a positive function $\psi_1(x)$ in $H^1_{rad}(\mathbb{R}^n)$. Let $\{v_k\}$ be a minimizing sequence for (2.2) with $j = 1$. Then $\{v_k\}$ is bounded in $H^1_{rad}(\mathbb{R}^n)$. Therefore, there exist a subsequence of $\{v_k\}$ (we still denote it by the same letter) and $v_0 \in H^1_{rad}(\mathbb{R}^n)$ such that $v_k \rightharpoonup v_0$ weakly in $H^1_{rad}(\mathbb{R}^n)$ and $v_k \to v_0$ strongly in $L^{p+1}_{rad}(\mathbb{R}^n)$. The last convergence is because of the compactness of the embedding $H^1_{rad}(\mathbb{R}^n) \hookrightarrow L^q_{rad}(\mathbb{R}^n)$ for $2 < q < 2 + 4/(n-2)$ (see [17]). We show that $v_0 \neq 0$. Suppose that $v_0 = 0$. Then, from $K_1(v_k) \leq 0$, together with the strong convergence $v_k \to 0$ in $L^{p+1}(\mathbb{R}^n)$, it follows that $v_k \to 0$ strongly in $H^1(\mathbb{R}^n)$. However, by $K_1(v_k) \leq 0$ and the Sobolev inequality, we have

$$\|\nabla v_k\|_2^2 + m^2\|v_k\|_2^2 \leq \|v_k\|_{p+1}^{p+1} \leq C_0(\|\nabla v_k\|_2^2 + m^2\|v_k\|_2^2)^{(p+1)/2}.$$

Since $v_k \neq 0$, we have $\|\nabla v_k\|_2^2 + m^2\|v_k\|_2^2 \geq C_0^{-2/(p-1)}$, which contradicts to the strong convergence $v_k \to 0$ in $H^1(\mathbb{R}^n)$. Thus, we see that $v_0 \in H^1(\mathbb{R}^n) \setminus \{0\}$. Therefore, by the lower semicontinuity of the norm in $H^1_{rad}(\mathbb{R}^n)$, together with the strong convergence $v_k \to 0$ in $L^{p+1}(\mathbb{R}^n)$, we have

$$K_1(v_0) \leq \liminf_{k\to\infty} K_1(v_k) \leq 0, \quad d_1 \leq J_1(v_0) \leq \liminf_{k\to\infty} J_1(v_k) = d_1.$$

Hence, $v_0$ attains the infimum of (2.2) for $j = 1$. Since $\psi_1 := |v_0|$ also attains (2.2) for $j = 1$, we see that (2.2) for $j = 1$ is attained at a positive function $\psi_1(x)$ in $H^1_{rad}(\mathbb{R}^n)$, and $K_1(\psi_1) = 0$ and $J(\psi_1) = d_1$. Next, we show that $\psi_1$ is a solution of (1.6). Since $\psi_1$ attains (2.1) for $j = 1$, there exists a Lagrange multiplier $\lambda_1 \in \mathbb{R}$ such that $J'(\psi_1) = \lambda_1 K_1'(\psi_1)$. Then, we have

$$\begin{aligned} 0 &= K_1(\psi_1) = \langle J'(\psi_1), \psi_1\rangle = \lambda_1\langle K_1'(\psi_1), \psi_1\rangle \\ &= \lambda_1\{2\|\nabla\psi_1\|_2^2 + 2m^2\|\psi_1\|_2^2 - (p+1)\|\psi_1\|_{p+1}^{p+1}\} \\ &= -(p-1)\lambda_1\|\psi_1\|_{p+1}^{p+1}, \end{aligned}$$

where in the last identity we used that $K_1(\psi_1) = 0$. This implies $\lambda_1 = 0$. So, we have $J'(\psi_1) = 0$, namely the positive function $\psi_1 \in H^1_{rad}(\mathbb{R}^n)$ is a solution of (1.6).

Finally, since the ground state $\psi(x)$ is the unique positive solution of (1.6) in $H^1_{rad}(\mathbb{R}^n)$, we have $\psi_1 = \psi_2 = \psi$. The identities $K_j(\psi) = K_j(\psi_j) = 0$ for $j = 1, 2$ lead to $d_j = J(\psi_j) = J(\psi)$ for $j = 1, 2$ which imply $d_1 = d_2$. □

Denote by $d = d_1 = d_2$ and $\Sigma = \Sigma_1 \cap \Sigma_2$, where

$$\Sigma_j = \{(u, v) \in H^1_{rad}(\mathbb{R}^n) \times L^2_{rad}(\mathbb{R}^n) : \; E(u, v) < d, \; K_j(u) < 0\}, \quad j = 1, 2. \quad (2.4)$$

Note that $(\lambda\psi, 0) \in \Sigma$ for any $\lambda > 1$.

**Lemma 4.** *The set $\Sigma$ is invariant under the flow of* (1.5). *That is, if $(u_0, u_1) \in \Sigma$, then the solution $u(t, x)$ of* (1.5) *with data $(u_0, u_1)$ satisfies $(u(t, .), \partial_t u(t, .)) \in \Sigma$ for any $t \in [0, T^*)$, where $T^*$ is the life span of the solution $u(t, x)$.*

*Proof.* It is enough to prove that the sets $\Sigma_j$ $(j = 1, 2)$ are invariant under the flow of (1.5). From the conservation of energy, we have $E(u(t), \partial_t u(t)) = E(u_0, u_1) < d$ for any $t \in [0, T^*)$. Thus, to conclude the proof, we have only to show that $K_j(u(t)) < 0$ for any $t \in [0, T^*)$. Suppose that there exists $t_0 \in (0, T^*)$ such that $K_j(u(t_0)) = 0$ and $K_j(u(t)) < 0$ for $t \in [0, t_0)$. Then, it follows from Lemma 3 that $J_j(u(t)) \geq d_j > 0$ for $t \in [0, t_0)$. Thus, we see that $u(t_0) \neq 0$. Since $K_j(u(t_0)) = 0$ and $u(t_0) \neq 0$, it follows from the definition of $d_j$ that $d_j \leq J(u(t_0)) \leq E(u(t_0), \partial_t u(t_0)) < d_j$, which is a contradiction. This completes the proof. □

For $\lambda > 1$, let $u_\lambda$ be the solution of (1.5) with data $(\lambda\psi, 0)$ where $\psi$ is the ground state of (1.6). Let $T_\lambda$ be the life span of $u_\lambda$. Denote by $E_\lambda$ and $Q_\lambda$ the energy and the charge of the solution $u_\lambda$ respectively. Let

$$I_\lambda(t) = \frac{1}{2}\|u_\lambda(t, .)\|_2^2, \quad 0 \leq t < T_\lambda.$$

The key lemma is the following lower estimate for the second derivative $I_\lambda''(t)$.

**Lemma 5.** *For any $\lambda > 1$, there exists a constant $a_\lambda > 0$ such that*

$$I_\lambda''(t) \geq \frac{p+3}{2}\|\partial_t u_\lambda(t, .)\|_2^2 + a_\lambda, \quad 0 \leq t < T_\lambda.$$

*Proof.* We have

$$I'_\lambda(t) = \operatorname{Re}\int_{\mathbb{R}^n} \partial_t u_\lambda(t,x)\overline{u_\lambda(t,x)}\,dx.$$

By standard approximation arguments we can prove that $I''_\lambda(t)$ exists in $[0,T_\lambda)$ and

$$\begin{aligned} I''_\lambda(t) &= \|\partial_t u_\lambda(t,.)\|_2^2 + \operatorname{Re}\int_{\mathbb{R}^n} \partial_t^2 u_\lambda(t,x)\overline{u_\lambda(t,x)}\,dx \\ &= \|\partial_t u_\lambda(t,.)\|_2^2 + 2\gamma \operatorname{Im}\int_{\mathbb{R}^n} \partial_t u_\lambda(t,x)\overline{u_\lambda(t,x)}\,dx - K_1(u_\lambda(t)). \end{aligned} \tag{2.5}$$

Since

$$2\gamma \operatorname{Im}\int_{\mathbb{R}^n} \partial_t u_\lambda(t,x)\overline{u_\lambda(t,x)}\,dx = 2\gamma Q_\lambda - 2\gamma^2\|u_\lambda(t,.)\|_2^2,$$

and

$$-K_1(u_\lambda(t)) = \frac{p+1}{2}\|\partial_t u_\lambda(t,.)\|_2^2 + (p+1)J_1(u_\lambda(t)) - (p+1)E_\lambda,$$

we obtain

$$I''_\lambda(t) = \frac{p+3}{2}\|\partial_t u_\lambda(t,.)\|_2^2 + (p+1)J_1(u_\lambda(t)) - 2\gamma^2\|u_\lambda(t,.)\|_2^2 - (p+1)E_\lambda + 2\gamma Q_\lambda. \tag{2.6}$$

Here, we note that for any $\lambda > 1$ we have

$$E_\lambda = J(\lambda\psi) < d, \quad \gamma Q_\lambda = \lambda^2\gamma^2\|\psi\|_2^2 > \gamma^2\|\psi\|_2^2.$$

By Lemma 3 it follows the identity $J_1(\psi) = J_2(\psi) = d$ which implies

$$\|\psi\|_2^2 = \frac{(n+2)-(n-2)p}{(p-1)m^2}d.$$

Thus, we have

$$\begin{aligned} &(p+1)E_\lambda - 2\gamma Q_\lambda < (p+1)d - 2\gamma^2\|\psi\|_2^2 \\ &= (\frac{p-1}{2} - \frac{2\gamma^2}{m^2})\frac{2(p+1)}{p-1}d + \frac{2\gamma^2}{m^2}nd. \end{aligned} \tag{2.7}$$

Since for $\lambda > 1$ the data $(\lambda\psi, 0)$ are in $\Sigma$, by Lemma 4 the solution $u_\lambda(t,x)$ of (1.5) with data $(\lambda\psi, 0)$ remains in $\Sigma$ for any $0 \le t < T_\lambda$. Because of the variational definition of $d = d_1 = d_2$ due to Lemma 3 we have

$$J_1(u_\lambda(t)) = \frac{p-1}{2(p+1)}(\|\nabla u_\lambda(t,.)\|_2^2 + m^2\|u_\lambda(t,.)\|_2^2) \ge d$$

and

$$J_2(u_\lambda(t)) = \frac{1}{n}\|\nabla u_\lambda(t,.)\|_2^2 \ge d$$

for any $0 \le t < T_\lambda$. Then we estimate the term $(p+1)J_1(u_\lambda(t)) - 2\gamma^2\|u_\lambda(t)\|_2^2$ in the right hand side of the identity (2.6) in a following way

$$\begin{aligned} &(p+1)J_1(u_\lambda(t)) - 2\gamma^2\|u_\lambda(t,.)\|_2^2 \\ &= \frac{p-1}{2}(\|\nabla u_\lambda(t,.)\|_2^2 + m^2\|u_\lambda(t,.)\|_2^2) - 2\gamma^2\|u_\lambda(t,.)\|_2^2 \\ &= (\frac{p-1}{2} - \frac{2\gamma^2}{m^2})(\|\nabla u_\lambda(t,.)\|_2^2 + m^2\|u_\lambda(t,.)\|_2^2) + \frac{2\gamma^2}{m^2}\|\nabla u_\lambda(t,.)\|_2^2 \\ &\ge (\frac{p-1}{2} - \frac{2\gamma^2}{m^2})\frac{2(p+1)}{p-1}d + \frac{2\gamma^2}{m^2}nd, \end{aligned} \tag{2.8}$$

where the assumption $4\gamma^2 \le (p-1)m^2$ was used. Finally, using the estimate (2.7) together with (2.8) we can rewrite (2.6) in the form

$$I_\lambda''(t) \ge \frac{p+3}{2}\|\partial_t u_\lambda(t,.)\|_2^2 + a_\lambda, \quad 0 \le t < T_\lambda,$$

where

$$a_\lambda = (\frac{p-1}{2} - \frac{2\gamma^2}{m^2})\frac{2(p+1)}{p-1}d + \frac{2\gamma^2}{m^2}nd - (p+1)E_\lambda + 2\gamma Q_\lambda > 0.$$

This completes the proof. □

Now the proof of Theorem 2 follows from Lemma 5 and concavity arguments due to Levine [9] as in Payne and Sattinger [13]. For the sake of completeness, we give the proof.

*Proof of Theorem 2.* Assume that the life span $T_\lambda = \infty$. By Lemma 5, we have $I_\lambda''(t) \ge a_\lambda > 0$ for any $t \in [0,\infty)$. This implies that there exists $t_1 \in (0,\infty)$ such that $I_\lambda'(t) > 0$ for any $t \in [t_1,\infty)$ and as well $I_\lambda(t) > 0$ for any $t \in [t_1,\infty)$. Let $\alpha = (p-1)/4$. Then by using Lemma 5 we obtain the following estimate

$$\begin{aligned} &I_\lambda''(t)I_\lambda(t) - (\alpha+1)I_\lambda'(t)^2 \\ &\ge \frac{p+3}{4}\left\{\|\partial_t u_\lambda(t)\|_2^2\|u_\lambda(t)\|_2^2 - \left(\mathrm{Re}\int_{\mathbb{R}^n}\partial_t u_\lambda(t,x)\overline{u_\lambda(t,x)}\,dx\right)^2\right\} \ge 0. \end{aligned}$$

Thus, for $t \in [t_1,\infty)$, we have

$$\begin{aligned} (I_\lambda(t)^{-\alpha})' &= -\alpha I_\lambda(t)^{-\alpha-1}I_\lambda'(t) < 0, \\ (I_\lambda(t)^{-\alpha})'' &= -\alpha I_\lambda(t)^{-\alpha-2}\{I_\lambda''(t)I_\lambda(t) - (\alpha+1)I_\lambda'(t)^2\} \le 0. \end{aligned}$$

Therefore,

$$I_\lambda(t)^{-\alpha} \le I_\lambda(t_1)^{-\alpha} - \alpha I_\lambda(t_1)^{-\alpha-1}I_\lambda'(t_1)(t-t_1), \quad t \in [t_1,\infty),$$

so there exists $t_2 \in (t_1,\infty)$ such that $I_\lambda(t_2)^{-\alpha} \le 0$. However, this is a contradiction. This completes the proof. □

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