

## REMARKS ON GLOBAL EXISTENCE AND BLOWUP FOR DAMPED NONLINEAR SCHRÖDINGER EQUATIONS

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**ABSTRACT.** We consider the Cauchy problem for the damped nonlinear Schrödinger equations, and prove some blowup and global existence results which depend on the size of the damping coefficient. We also discuss the  $L^2$  concentration phenomenon of blowup solutions in the critical case.

**1. Introduction.** In this paper, we study global existence and blowup of solutions to the Cauchy problem for the damped nonlinear Schrödinger equation:

$$i\partial_t u + \Delta u + |u|^{p-1}u + iau = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n \quad (1.1)$$

with initial data  $u(0) = u_0 \in H^1(\mathbb{R}^n)$ , where  $a \geq 0$ ,  $p > 1$ , and  $p < 1 + 4/(n - 2)$  if  $n \geq 3$ . The equation (1.1) arises in various areas of nonlinear optics, plasma physics and fluid mechanics, and has been studied by many mathematicians and physicists (see, e.g., [4, 10, 25, 26] and references therein).

It is known that the Cauchy problem for (1.1) is locally well-posed in  $H^1(\mathbb{R}^n)$  (see Kato [12] and also Cazenave [2, Section 4.4]): For any  $u_0 \in H^1(\mathbb{R}^n)$ , there exist  $T_a^*(u_0) \in (0, \infty]$  and a unique solution  $u(t)$  of (1.1) with  $u(0) = u_0$  such that  $u \in C([0, T_a^*(u_0)); H^1(\mathbb{R}^n))$ . Moreover,  $T_a^*(u_0)$  is the maximal existence time of the solution  $u(t)$  in the sense that if  $T_a^*(u_0) < \infty$  then  $\lim_{t \rightarrow T_a^*(u_0)} \|u(t)\|_{H^1} = \infty$ .

First, we recall some known results for the case  $a = 0$  (see [2, 23] for more information). When  $p < 1 + 4/n$ , we have  $T_0^*(u_0) = \infty$  for any  $u_0 \in H^1(\mathbb{R}^n)$ . When  $p \geq 1 + 4/n$ , we have  $T_0^*(u_0) = \infty$  if the initial data  $u_0$  is sufficiently small in  $H^1(\mathbb{R}^n)$ , and  $T_0^*(u_0) < \infty$  if  $u_0 \in \Sigma$  and  $E(u_0) < 0$ , where we put

$$\Sigma = \{v \in H^1(\mathbb{R}^n) : xv \in L^2(\mathbb{R}^n)\}$$

and the energy  $E$  is defined by

$$E(v) = \frac{1}{2} \|\nabla v\|_{L^2}^2 - \frac{1}{p+1} \|v\|_{L^{p+1}}^{p+1} \quad (1.2)$$

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(see [6, 9, 24, 29] and also Remark 1.3 below). The global existence result follows from the local well-posedness in  $H^1(\mathbb{R}^n)$ , the conservation laws of energy  $E(u)$  and charge  $\|u\|_{L^2}^2$ , and the Gagliardo-Nirenberg inequality:

$$\|v\|_{L^{p+1}}^{p+1} \leq C \|v\|_{L^2}^{(p+1)-n(p-1)/2} \|\nabla v\|_{L^2}^{n(p-1)/2}, \quad v \in H^1(\mathbb{R}^n). \quad (1.3)$$

The blowup result is based on the virial identity:

$$\frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 16P(u(t)), \quad t \in [0, T_0^*(u_0)),$$

where

$$P(v) = \frac{1}{2} \|\nabla v\|_2^2 - \frac{n(p-1)}{4(p+1)} \|v\|_{p+1}^{p+1}. \quad (1.4)$$

Note that  $P(v) \leq E(v)$  for all  $v \in H^1(\mathbb{R}^n)$  if  $p \geq 1 + 4/n$ .

Next, we consider the case  $a > 0$  in (1.1). In this case, we have

$$\|u(t)\|_{L^2} = e^{-at} \|u_0\|_{L^2}, \quad t \in [0, T_a^*(u_0)), \quad (1.5)$$

but the energy  $E(u(t))$  is no longer conserved nor decreasing. In fact, we have

$$\frac{d}{dt} E(u(t)) = -aK(u(t)), \quad t \in [0, T_a^*(u_0)), \quad (1.6)$$

where

$$K(v) = \|\nabla v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1} \quad (1.7)$$

(see M. Tsutsumi [25]). It is proved in [25] that when  $p > 1 + 4/n$ , we have  $T_a^*(u_0) < \infty$  if  $u_0 \in \Sigma$  satisfies

$$E(u_0) \leq 0 \quad \text{and} \quad \frac{(p-1)a}{(p-1)n-4} I(u_0) + V(u_0) < 0, \quad (1.8)$$

where we put

$$I(v) = \|xv\|_{L^2}^2, \quad V(v) = \Im \int_{\mathbb{R}^n} x \cdot \nabla v(x) \overline{v(x)} dx.$$

The proof in [25] is based on the following two identities:

$$\frac{d}{dt} I(u(t)) + 2aI(u(t)) = 4V(u(t)), \quad (1.9)$$

$$\frac{d}{dt} V(u(t)) + 2aV(u(t)) = 4P(u(t)) \quad (1.10)$$

for  $t \in [0, T_a^*(u_0))$ , where  $P$  is the functional defined by (1.4).

For the case  $p < 1 + 4/n$ , we have  $T_a^*(u_0) = \infty$  for all  $u_0 \in H^1(\mathbb{R}^n)$  and for all  $a \geq 0$ . Indeed, for this case, we have the following blowup alternative: if  $T_a^*(u_0) < \infty$  then  $\lim_{t \rightarrow T_a^*(u_0)} \|u(t)\|_{L^2} = \infty$  (see [27], and also Sections 4.6 and 5.2 of [2]). This alternative and (1.5) imply that  $T_a^*(u_0) = \infty$  for all  $u_0 \in H^1(\mathbb{R}^n)$  (for the long time behavior of global solutions for (1.1) with external force  $f(x)$ , see, e.g., [5, 11, 14]). Therefore, we consider the case  $p \geq 1 + 4/n$  only.

Now, we state our main results.

**Theorem 1.** *Assume that  $1 + 4/n \leq p < 1 + 4/(n-2)$ . For any  $u_0 \in H^1(\mathbb{R}^n)$  there exists  $a^*(\|u_0\|_{H^1}) \in [0, \infty)$  such that  $T_a^*(u_0) = \infty$  for all  $a \geq a^*(\|u_0\|_{H^1})$ .*

**Theorem 2.** *Let  $1 + 4/n < p < 1 + 4/(n - 2)$ . Assume that  $u_0 \in \Sigma$  satisfies one of the following conditions:*

- (i)  $E(u_0) < 0$ ,
- (ii)  $E(u_0) = 0$  and  $V(u_0) < 0$ ,
- (iii)  $E(u_0) > 0$  and  $V(u_0) < -\sqrt{2E(u_0)I(u_0)}$ .

*Then, there exists  $a_*(u_0) > 0$  such that  $T_a^*(u_0) < \infty$  for all  $a \in [0, a_*(u_0))$ .*

**Remark 1.1** A similar result to Theorem 1 is stated in [23, p.98] without proof (see also [4, Theorem 2.4]).

**Remark 1.2** For the case  $p = 1 + 4/n$  and  $a > 0$ , numerical simulations suggest the existence of finite time blowup solutions of (1.1) (see Fibich [4]), but it is an open problem whether there exist finite time blowup solutions for this case (see [20] for comparison with other types of dissipation).

**Remark 1.3** When  $a = 0$  and  $1 + 4/n \leq p < 1 + 4/(n - 2)$ , it is well known that we have  $T_0^*(u_0) < \infty$  if  $u_0 \in \Sigma$  satisfies one of the conditions (i), (ii) and

- (iii')  $E(u_0) > 0$  and  $V(u_0) \leq -\sqrt{2E(u_0)I(u_0)}$

(see [9, 24, 29, 2, 23]). We note that the sufficient condition (ii) in Theorem 2 follows from (1.8), but (i) and (iii) do not.

**Remark 1.4** For the case  $a = 0$ , it is proved by Cazenave and Weissler [3, Corollary 2.5] that when  $p_0(n) < p < 1 + 4/(n - 2)$ , for any  $u_0 \in \Sigma$ , there exists  $b^*(u_0) \in [0, \infty)$  such that  $T_0^*(e^{ib|x|^2}u_0) = \infty$  for all  $b \geq b^*(u_0)$ , where

$$p_0(n) = \frac{n + 2 + \sqrt{n^2 + 12n + 4}}{2n}. \quad (1.11)$$

Note that  $1 < p_0(n) < 1 + 4/n$ . The proof in [3] is based on the Strichartz estimates but not on the energy method. Since the energy  $E(u(t))$  is not conserved nor decreasing when  $a > 0$ , the energy method is not useful for the proof of Theorem 1, and we will apply the argument in the proofs of Proposition 2.4 and Corollary 2.5 of [3].

The rest of the paper is organized as follows. In Section 2, we give the proof of Theorem 1 along the argument in Cazenave and Weissler [3]. In Section 3, we prove Theorem 2 by modifying the proof of M. Tsutsumi [25]. In Section 4, we construct some invariant sets under the flow of (1.1), which are independent of the damping coefficient  $a \geq 0$ , and prove a global existence result for solutions with initial data in these invariant sets. In Section 5, assuming the existence of blowup solutions, we study the  $L^2$  concentration phenomenon for blowup solutions of (1.1) for the case  $p = 1 + 4/n$  and  $a > 0$ .

**2. Proof of Theorem 1.** In this section, we prove Theorem 1. Let  $U_a(t)$  be the propagator for the linear equation:

$$i\partial_t u + \Delta u + iau = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n. \quad (2.1)$$

The Cauchy problem for (1.1) with  $u(0) = u_0 \in H^1(\mathbb{R}^n)$  is equivalent to the integral equation (see [2, 12]):

$$u(t) = U_a(t)u_0 + i \int_0^t U_a(t-s)|u(s)|^{p-1}u(s) ds. \quad (2.2)$$

The following result is a key moment in the proof of Theorem 1.

**Proposition 3.** *Assume  $a > 0$  and  $1 + 4/n \leq p < 1 + 4/(n-2)$ . Then, there exists a positive constant  $\varepsilon$  independent of  $a$  such that if  $u_0 \in H^1(\mathbb{R}^n)$  and  $\|U_a(\cdot)u_0\|_{L^\theta(0,\infty;L^{p+1})} \leq \varepsilon$ , then  $T_a^*(u_0) = \infty$ , where*

$$\theta = \frac{2(p-1)(p+1)}{4 - (n-2)(p-1)}. \quad (2.3)$$

We begin with several space-time estimates, which will be used in the proof of Proposition 3. These estimates go back to Strichartz [22], and have been widely used to study nonlinear Schrödinger equations (see, e.g., [7, 12, 2, 23]). Remark that when  $a > 0$ ,  $U_a(t)$  are not unitary on  $L^2(\mathbb{R}^n)$ , and the expression (2.4) below plays an important role, especially in the proof of Lemma 6.

We define

$$\Phi_a[f](t) = \int_0^t U_a(t-s)f(s) ds$$

for  $t > 0$ . Note that

$$\Phi_a[f](t) = e^{-at} \int_0^t U_0(t-s)e^{as}f(s) ds. \quad (2.4)$$

Moreover, for  $t > 0$  and  $\tau \in \mathbb{R}$ , we define

$$\Psi_a[f](t, \tau) = e^{-at} \int_0^t U_0(\tau-s)e^{as}f(s) ds.$$

We say that a pair  $(q, r)$  is admissible if  $2 \leq r < \infty$ ,  $2 < q \leq \infty$  and

$$\frac{2}{q} = n \left( \frac{1}{2} - \frac{1}{r} \right).$$

We do not consider the endpoint case  $(q, r) = (2, 2n/(n-2))$  for  $n \geq 3$ .

**Lemma 4.** *Let  $0 < T \leq \infty$  and  $2 \leq r \leq \infty$ . For  $0 < t < T$  and  $\tau \in \mathbb{R}$ , we have*

$$\|\Psi_a[f](t, \tau)\|_{L^r} \leq C \int_0^T |\tau - s|^{-n(1/2-1/r)} \|f(s)\|_{L^{r'}} ds,$$

where  $C$  depends only on  $n$  and  $r$ , and  $r'$  denotes the Hölder conjugate exponent of  $r$ .

*Proof.* By the decay estimates for  $U_0(t)$  and by (2.4), we have

$$\begin{aligned} \|\Psi_a[f](t, \tau)\|_{L^r} &\leq e^{-at} \int_0^t \|U_0(\tau-s)e^{as}f(s)\|_{L^r} ds \\ &\leq C e^{-at} \int_0^t |\tau-s|^{-n(1/2-1/r)} \|e^{as}f(s)\|_{L^{r'}} ds \\ &\leq C \int_0^t |\tau-s|^{-n(1/2-1/r)} \|f(s)\|_{L^{r'}} ds. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 5.** *Let  $0 < T \leq \infty$  and  $0 < t < T$ . For any admissible pair  $(q, r)$ , we have*

$$\begin{aligned} \|\Phi_a[f]\|_{L^q(0,T;L^r)} &\leq C \|f\|_{L^{q'}(0,T;L^{r'})}, \\ \|\Psi_a[f](t, \cdot)\|_{L^q(0,T;L^r)} &\leq C \|f\|_{L^{q'}(0,T;L^{r'})}, \end{aligned}$$

where  $C$  depends only on  $n$  and  $r$ .

*Proof.* For  $0 < t < T$  and  $\tau \in \mathbb{R}$ , it follows from Lemma 4 that

$$\|\Psi_a[f](t, \tau)\|_{L^r} \leq C \int_0^T |\tau - s|^{-2/q} \|f(s)\|_{L^{r'}} ds.$$

Thus, the result follows from the Riesz potential inequalities.  $\square$

**Lemma 6.** *Let  $0 < T \leq \infty$ . For any admissible pair  $(q, r)$ , we have*

$$\|\Phi_a[f]\|_{L^\infty(0, T; L^2)} \leq C \|f\|_{L^{q'}(0, T; L^{r'})},$$

where  $C$  depends only on  $n$  and  $r$ .

*Proof.* For  $0 < t < T$ , by (2.4) we have

$$\begin{aligned} & \|\Phi_a[f](t)\|_{L^2}^2 \\ &= \left( e^{-at} \int_0^t U_0(t-s) e^{as} f(s) ds, e^{-at} \int_0^t U_0(t-\sigma) e^{a\sigma} f(\sigma) d\sigma \right)_{L^2} \\ &= e^{-2at} \int_0^t \int_0^t (U_0(t-s) e^{as} f(s), U_0(t-\sigma) e^{a\sigma} f(\sigma))_{L^2} d\sigma ds \\ &= e^{-2at} \int_0^t \int_0^t (e^{as} f(s), U_0(s-\sigma) e^{a\sigma} f(\sigma))_{L^2} d\sigma ds \\ &= e^{-at} \int_0^t (e^{as} f(s), \Psi_a[f](t, s))_{L^2} ds. \end{aligned}$$

By the Hölder inequality and Lemma 5, we have

$$\begin{aligned} \|\Phi_a[f](t)\|_{L^2}^2 &\leq e^{-at} \int_0^t \|e^{as} f(s)\|_{L^{r'}} \|\Psi_a[f](t, s)\|_{L^r} ds \\ &\leq \int_0^t \|f(s)\|_{L^{r'}} \|\Psi_a[f](t, s)\|_{L^r} ds \\ &\leq \|f\|_{L^{q'}(0, T; L^{r'})} \|\Psi_a[f](t, \cdot)\|_{L^q(0, T; L^r)} \leq C \|f\|_{L^{q'}(0, T; L^{r'})}^2 \end{aligned}$$

for  $t \in (0, T)$ . This completes the proof.  $\square$

**Lemma 7.** *Let  $0 < T \leq \infty$ ,  $2 < r < 2n/(n-2)$ , and let  $\theta, \tilde{\theta} \in (1, \infty)$  satisfy*

$$\frac{1}{\tilde{\theta}} + \frac{1}{\theta} = n \left( \frac{1}{2} - \frac{1}{r} \right). \quad (2.5)$$

Then, we have

$$\|\Phi_a[f]\|_{L^\theta(0, T; L^r)} \leq C \|f\|_{L^{\tilde{\theta}'}(0, T; L^{r'})},$$

where  $C$  depends only on  $n$ ,  $r$  and  $\theta$ .

*Proof.* For  $0 < t < T$ , it follows from Lemma 4 that

$$\|\Phi_a[f](t)\|_{L^r} \leq C \int_0^T |t-s|^{-n(1/2-1/r)} \|f(s)\|_{L^{r'}} ds.$$

Thus, the result follows from the Riesz potential inequalities.  $\square$

Now we are in a position to prove Proposition 3.

*Proof of Proposition 3.* We follow the proof of Proposition 2.4 in [3]. Let  $r = p+1$  and  $(q, r)$  be the corresponding admissible pair. Let  $\theta$  be given by (2.3), and  $\tilde{\theta}$  satisfy (2.5). Since  $p \geq 1 + 4/n > p_0(n)$  (see (1.11)), we have  $\theta, \tilde{\theta} \in (q/2, \infty)$ . Let  $\varepsilon > 0$  and let  $u_0 \in H^1(\mathbb{R}^n)$  satisfy  $\|U_a(\cdot)u_0\|_{L^\theta(0, \infty; L^r)} \leq \varepsilon$ , and  $u(t)$  be the solution

of (1.1) with  $u(0) = u_0$ . We denote  $T_a^*(u_0)$  by  $T^*$  for simplicity. Since  $pr' = r$ , by Lemma 7, we have

$$\begin{aligned} \|u\|_{L^\theta(0,T;L^r)} &\leq \|U_a(\cdot)u_0\|_{L^\theta(0,T;L^r)} + \|\Phi_a[|u|^{p-1}u]\|_{L^\theta(0,T;L^r)} \\ &\leq \varepsilon + C_1\||u|^{p-1}u\|_{L^{\theta'}(0,T;L^{r'})} = \varepsilon + C_1\|u\|_{L^\theta(0,T;L^r)}^p \end{aligned} \quad (2.6)$$

for any  $T \in (0, T^*)$ , where  $C_1 > 0$  is a constant independent of  $u_0$ . Moreover, since  $p\tilde{\theta}' = \theta$  and  $1/q' = 1/q + (p-1)/\theta$ , we have

$$\||u|^{p-1}u\|_{L^{q'}(0,T;W^{1,r'})} \leq C\|u\|_{L^\theta(0,T;L^r)}^{p-1}\|u\|_{L^q(0,T;W^{1,r})}$$

and by Lemma 5, we have

$$\begin{aligned} \|u\|_{L^q(0,T;W^{1,r})} &\leq \|U_a(\cdot)u_0\|_{L^q(0,T;W^{1,r})} + \|\Phi_a[|u|^{p-1}u]\|_{L^q(0,T;W^{1,r})} \\ &\leq \|U_0(\cdot)u_0\|_{L^q(0,T;W^{1,r})} + C\||u|^{p-1}u\|_{L^{q'}(0,T;W^{1,r'})} \\ &\leq C_2\|u_0\|_{H^1} + C_2\|u\|_{L^\theta(0,T;L^r)}^{p-1}\|u\|_{L^q(0,T;W^{1,r})} \end{aligned} \quad (2.7)$$

for any  $T \in (0, T^*)$ , where  $C_2 > 0$  is a constant independent of  $u_0$ . Put  $C_0 = \max\{C_1, C_2\}$ , and assume that  $\varepsilon$  satisfies  $2^p C_0 \varepsilon^{p-1} < 1$ . Since  $\|u\|_{L^\theta(0,T;L^r)}$  depends continuously on  $T$ , it follows from (2.6) that

$$\|u\|_{L^\theta(0,T^*;L^r)} \leq 2\varepsilon. \quad (2.8)$$

By (2.7) and (2.8), we have

$$\|u\|_{L^q(0,T^*;W^{1,r})} \leq 2C_0\|u_0\|_{H^1}. \quad (2.9)$$

Finally, by Lemma 6 and estimates (2.8) and (2.9), we obtain

$$\begin{aligned} \|u\|_{L^\infty(0,T^*;H^1)} &\leq \|U_a(\cdot)u_0\|_{L^\infty(0,T^*;H^1)} + \|\Phi_a[|u|^{p-1}u]\|_{L^\infty(0,T^*;H^1)} \\ &\leq \|U_0(\cdot)u_0\|_{L^\infty(0,T^*;H^1)} + C\||u|^{p-1}u\|_{L^{q'}(0,T^*;W^{1,r'})} \\ &\leq C\|u_0\|_{H^1} + C\|u\|_{L^\theta(0,T^*;L^r)}^{p-1}\|u\|_{L^q(0,T^*;W^{1,r})} < \infty, \end{aligned}$$

which implies  $T^* = \infty$  by the blowup alternative (see, e.g., [2, Theorem 4.4.1]). This completes the proof.  $\square$

*Proof of Theorem 1.* Since  $U_a(t) = e^{-at}U_0(t)$ , we have

$$\|U_a(\cdot)u_0\|_{L^\theta(0,\infty;L^{p+1})}^\theta = \int_0^\infty e^{-\theta at} \|U_0(t)u_0\|_{L^{p+1}}^\theta dt.$$

By the Sobolev inequality, we have

$$\|U_0(t)u_0\|_{L^{p+1}} \leq C\|U_0(t)u_0\|_{H^1} = C\|u_0\|_{H^1}$$

for all  $t \geq 0$ . Therefore, we obtain

$$\lim_{a \rightarrow \infty} \|U_a(\cdot)u_0\|_{L^\theta(0,\infty;L^{p+1})} = 0. \quad (2.10)$$

Theorem 1 follows from Proposition 3 and (2.10).  $\square$

**3. Proof of Theorem 2.** In this section, we give the proof of Theorem 2 by modifying the proof of M. Tsutsumi [25].

*Proof of Theorem 2.* Let  $u_0 \in \Sigma$  satisfy one of the conditions (i), (ii) and (iii), and assume that the solution  $u(t)$  of (1.1) with  $u(0) = u_0$  exists for all  $t \in [0, \infty)$ . We put  $E(t) = E(u(t))$ ,  $I(t) = I(u(t))$ ,  $V(t) = V(u(t))$ ,  $K(t) = K(u(t))$  and  $P(t) = P(u(t))$ . Then, by (1.6), (1.9) and (1.10), we have

$$E'(t) = -aK(t), \quad (3.1)$$

$$I'(t) + 2aI(t) = 4V(t), \quad (3.2)$$

$$V'(t) + 2aV(t) = 4P(t) \quad (3.3)$$

for all  $t \geq 0$ . Define

$$b = \frac{(n+2) - (n-2)p}{n(p-1) - 4} 2a. \quad (3.4)$$

Then, since  $1 + 4/n < p < (n+2)/(n-2)$ , we have  $b > 0$ , and by (3.1) and (3.4) and by the definitions (1.2), (1.4) and (1.7) of  $E$ ,  $P$  and  $K$ , we have

$$\frac{d}{dt}\{e^{-bt}E(t)\} = -(2a+b)e^{-bt}P(t). \quad (3.5)$$

Moreover, by (3.2) and (3.3), we have

$$\frac{d}{dt}\{e^{-bt}I(t)\} = -(2a+b)e^{-bt}I(t) + 4e^{-bt}V(t), \quad (3.6)$$

$$\frac{d}{dt}\{e^{-bt}V(t)\} = -(2a+b)e^{-bt}V(t) + 4e^{-bt}P(t). \quad (3.7)$$

Since  $P(t) \leq E(t)$ , by (3.5), we have

$$e^{-bt}P(t) \leq e^{-bt}E(t) = E(0) - (2a+b) \int_0^t e^{-bs}P(s) ds. \quad (3.8)$$

Here, we put

$$\gamma = 2a + b = \frac{4(p-1)}{n(p-1) - 4} a \quad (3.9)$$

and

$$\tilde{P}(t) = \int_0^t e^{-bs}P(s) ds, \quad \tilde{V}(t) = \int_0^t e^{-bs}V(s) ds.$$

Then, by (3.8) we have

$$\tilde{P}'(t) + \gamma\tilde{P}(t) \leq E(0).$$

Since  $\tilde{P}(0) = 0$ , we have

$$e^{\gamma t}\tilde{P}(t) \leq \frac{E(0)}{\gamma}(e^{\gamma t} - 1). \quad (3.10)$$

By (3.7), we have

$$\tilde{V}'(t) + \gamma\tilde{V}(t) = V(0) + 4\tilde{P}(t),$$

and by (3.10), we have

$$\frac{d}{dt}\{e^{\gamma t}\tilde{V}(t)\} = V(0)e^{\gamma t} + 4e^{\gamma t}\tilde{P}(t) \leq V(0)e^{\gamma t} + \frac{4E(0)}{\gamma}(e^{\gamma t} - 1).$$

Since  $\tilde{V}(0) = 0$ , we have

$$e^{\gamma t}\tilde{V}(t) \leq \frac{V(0)}{\gamma}(e^{\gamma t} - 1) + \frac{4E(0)}{\gamma^2}(e^{\gamma t} - 1 - \gamma t). \quad (3.11)$$

Moreover, since  $\gamma = 2a + b > 0$  and  $I(t) = \|xu(t)\|_{L^2}^2 \geq 0$ , by (3.6) and (3.11), we have

$$e^{2at}I(t) = e^{\gamma t}e^{-bt}I(t) \leq e^{\gamma t}\{I(0) + 4\tilde{V}(t)\} \leq g(\gamma, t), \quad (3.12)$$

where we put

$$g(\gamma, t) = I(0)e^{\gamma t} + \frac{4V(0)}{\gamma}(e^{\gamma t} - 1) + \frac{16E(0)}{\gamma^2}(e^{\gamma t} - 1 - \gamma t).$$

By the Taylor expansion, for each  $t > 0$ , we have

$$g(\gamma, t) = I(0) + 4V(0)t + 8E(0)t^2 + O(\gamma) \quad (3.13)$$

as  $\gamma \rightarrow 0$ . Since we assume that  $u_0$  satisfies one of the conditions (i), (ii) and (iii), there exists  $t_0 = t_0(u_0) \in (0, \infty)$  such that

$$I(0) + 4V(0)t_0 + 8E(0)t_0^2 < 0. \quad (3.14)$$

Note that  $t_0$  is independent of  $\gamma$ . Since  $\gamma$  is proportional to  $a$  by (3.9), it follows from (3.13) and (3.14) that there exists  $a_*(u_0) > 0$  such that  $g(\gamma, t_0) < 0$  for all  $a \in [0, a_*(u_0))$ . Thus, by (3.12), we see that  $I(t_0) < 0$  for all  $a \in [0, a_*(u_0))$ . However, since  $I(t) = \|xu(t)\|_{L^2}^2 \geq 0$  for all  $t \geq 0$ , this is a contradiction. Therefore, we conclude that  $T_a^*(u_0) < \infty$  for all  $a \in [0, a_*(u_0))$ .  $\square$

**4. Invariant sets and global existence.** In this section, we construct some invariant sets under the flow of (1.1), which are independent of the damping coefficient  $a \geq 0$ , and prove that the solutions of (1.1) are global if the initial data belong to the invariant sets.

Let  $1 < p < 1 + 4/(n - 2)$ . For  $\omega > 0$ , we put

$$S_\omega(v) = E(v) + \frac{\omega}{2}\|v\|_{L^2}^2 = \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1}, \quad (4.1)$$

$$\begin{aligned} K_\omega(v) &= K(v) + \omega\|v\|_{L^2}^2 = \|\nabla v\|_{L^2}^2 + \omega\|v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1}, \\ d(\omega) &= \inf\{S_\omega(v) : v \in H^1(\mathbb{R}^n) \setminus \{0\}, K_\omega(v) = 0\}, \\ \mathcal{A}_\omega &= \{v \in H^1(\mathbb{R}^n) : S_\omega(v) < d(\omega), K_\omega(v) > 0\}. \end{aligned} \quad (4.2)$$

Note that since

$$S_\omega(v) - \frac{1}{p+1}K_\omega(v) = \frac{p-1}{2(p+1)}(\|\nabla v\|_{L^2}^2 + \omega\|v\|_{L^2}^2), \quad (4.3)$$

we have

$$d(\omega) = \frac{p-1}{2(p+1)} \inf\{\|\nabla v\|_{L^2}^2 + \omega\|v\|_{L^2}^2 : v \in H^1(\mathbb{R}^n) \setminus \{0\}, K_\omega(v) = 0\} \geq 0.$$

It is well-known that  $d(\omega) > 0$  and it is attained by a positive solution  $\varphi \in H^1(\mathbb{R}^n)$  of the stationary problem for (1.1) with  $a = 0$ :

$$-\Delta\varphi + \omega\varphi - |\varphi|^{p-1}\varphi = 0, \quad x \in \mathbb{R}^n \quad (4.4)$$

(see, e.g., [19]). Although the energy  $E(u(t))$  is not monotone for the case  $a > 0$  in general, we have the following global existence result.

**Theorem 8.** *Let  $1 < p < 1 + 4/(n - 2)$  and  $a \geq 0$ . If the data  $u_0 \in \bigcup_{\omega>0} \mathcal{A}_\omega$ , we have  $T_a^*(u_0) = \infty$ .*

The following lemma plays an important role in the proof of Theorem 8.



**Lemma 9.** *Let  $1 < p < 1 + 4/(n - 2)$  and  $a \geq 0$ . For any  $\omega > 0$ , the set  $\mathcal{A}_\omega$  is invariant under the flow of (1.1), i.e., if  $u_0 \in \mathcal{A}_\omega$  and  $u(t)$  is the solution of (1.1) with  $u(0) = u_0$ , then  $u(t) \in \mathcal{A}_\omega$  for all  $t \in [0, T_a^*(u_0))$ .*

*Proof.* By (1.5) and (1.6), we have

$$\frac{d}{dt} S_\omega(u(t)) = -a K_\omega(u(t)), \quad t \in [0, T_a^*(u_0)). \quad (4.5)$$

Since  $u \in C([0, T_a^*(u_0)), H^1(\mathbb{R}^n))$  and  $u_0 \in \mathcal{A}_\omega$ , we see that  $u(t) \in \mathcal{A}_\omega$  for small  $t \geq 0$ . Suppose that there exists  $t_1 \in (0, T_a^*(u_0))$  such that  $u(t) \in \mathcal{A}_\omega$  for all  $t \in [0, t_1)$ , and  $u(t_1) \notin \mathcal{A}_\omega$ . Then, we see that  $K_\omega(u(t)) \geq 0$  for all  $t \in [0, t_1]$ , and by (4.5), we have  $S_\omega(u(t_1)) \leq S_\omega(u_0) < d(\omega)$ . Since  $u(t_1) \notin \mathcal{A}_\omega$ , we have  $K_\omega(u(t_1)) = 0$ . Moreover, since  $u(t_1) \neq 0$ , by the definition of  $d(\omega)$ , we have  $d(\omega) \leq S_\omega(u(t_1))$ . This is a contradiction. Hence, we conclude that  $u(t) \in \mathcal{A}_\omega$  for all  $t \in [0, T_a^*(u_0))$ .  $\square$

*Proof of Theorem 8.* Let  $u_0 \in \mathcal{A}_\omega$  for some  $\omega > 0$ , and  $u(t)$  be the solution of (1.1) with  $u(0) = u_0$ . By Lemma 9, we see that  $S_\omega(u(t)) < d(\omega)$  and  $K_\omega(u(t)) > 0$  for all  $t \in [0, T_a^*(u_0))$ . By (4.3), we have

$$\begin{aligned} & \frac{p-1}{2(p+1)} (\|\nabla u(t)\|_{L^2}^2 + \omega \|u(t)\|_{L^2}^2) \\ &= S_\omega(u(t)) - \frac{1}{p+1} K_\omega(u(t)) \leq S_\omega(u(t)) \leq d(\omega) \end{aligned}$$

for all  $t \in [0, T_a^*(u_0))$ , which implies  $T_a^*(u_0) = \infty$ .  $\square$

**Remark 4.1** It seems that Theorem 1 does not follow from Theorem 8 and a simple scaling argument with  $a$  and  $\omega$ .

**Remark 4.2** For the case  $a = 0$  and  $p > 1 + 4/n$ , it is known that the sets

$$\begin{aligned} \mathcal{B}_\omega^+ &= \{v \in H^1(\mathbb{R}^n) : S_\omega(v) < d_1(\omega), P(v) > 0\}, \\ \mathcal{B}_\omega^- &= \{v \in H^1(\mathbb{R}^n) : S_\omega(v) < d_1(\omega), P(v) < 0\} \end{aligned}$$

are invariant under the flow of (1.1) with  $a = 0$ , where  $\omega > 0$ ,  $S_\omega$  and  $P$  are defined by (4.1) and (1.4) respectively, and

$$d_1(\omega) = \inf\{S_\omega(v) : v \in H^1(\mathbb{R}^n) \setminus \{0\}, P(v) = 0\}$$

(see [1, 2]). Note that

$$S_\omega(v) - \frac{4}{n(p-1)} P(v) = \frac{n(p-1)-4}{2n(p-1)} \|\nabla v\|_{L^2}^2 + \frac{\omega}{2} \|v\|_{L^2}^2,$$

and it is known that  $d_1(\omega) > 0$  and it is attained by a positive solution of (4.4) (see [1, 2]). Since the positive solution of (4.4) is unique, up to translations (see [13]), we see that  $d_1(\omega) = d(\omega)$ , where  $d(\omega)$  is defined by (4.2). Moreover, it is also known that we have  $T_0^*(u_0) = \infty$  if  $u_0 \in \mathcal{B}_\omega^+$ , and  $T_0^*(u_0) < \infty$  if  $u_0 \in \mathcal{B}_\omega^- \cap \Sigma$  (see [1, 2]). We do not know whether  $\mathcal{A}_\omega$  can be replaced by  $\mathcal{B}_\omega^+$  in Theorem 8 for the case  $a > 0$ , nor whether the conditions (i), (ii) and (iii) can be replaced by  $u_0 \in \mathcal{B}_\omega^-$  in Theorem 2.

5.  **$L^2$  concentration for critical case.** In this section, we study the  $L^2$  concentration phenomenon for blowup solutions of (1.1) for the case  $p = 1 + 4/n$  and  $a > 0$ . For the case  $a = 0$ , this phenomenon has been studied by many authors (see, e.g., [15, 16, 17, 28, 30, 31]). Because of Remark 1.2, we assume the existence of blowup solutions of (1.1) in the case  $p = 1 + 4/n$  and  $a > 0$ .

Let  $Q \in H^1(\mathbb{R}^n)$  be the ground state (unique positive radially symmetric solution) of

$$-\Delta Q + Q - |Q|^{4/n}Q = 0, \quad x \in \mathbb{R}^n \quad (5.1)$$

(see Kwong [13] for uniqueness). We state the following result.

**Theorem 10.** *Let  $p = 1 + 4/n$ ,  $n \geq 2$  and  $a > 0$ . Assume that  $u_0 \in H^1(\mathbb{R}^n)$  is radially symmetric, and suppose that the solution  $u(t)$  of (1.1) with  $u(0) = u_0$  blows up in finite time  $T \in (0, \infty)$ . Then, for any function  $\rho(t)$  satisfying  $\rho(t) \rightarrow \infty$  as  $t \rightarrow T$ , we have*

$$\limsup_{t \rightarrow T} \|u(t)\|_{L^2(\{|x| < \rho(t)\} / \|\nabla u(t)\|_{L^2})} \geq \|Q\|_{L^2}. \quad (5.2)$$

**Remark 5.1** By (1.5) and the blowup alternative, we have  $\lim_{t \rightarrow T} \|\nabla u(t)\|_{L^2} = \infty$  in Theorem 10. In particular, taking  $\rho(t) = R\|\nabla u(t)\|_{L^2}$  in (5.2), for any  $R > 0$  we have

$$\limsup_{t \rightarrow T} \|u(t)\|_{L^2(\{|x| < R\})} \geq \|Q\|_{L^2}.$$

**Remark 5.2** For the case  $a = 0$ , it is well-known that  $\limsup_{t \rightarrow T}$  can be replaced by  $\liminf_{t \rightarrow T}$  in (5.2) (see Y. Tsutsumi [28]). For related results on the Zakharov system, we refer to Gnanou and Merle [8, Theorem 1].

**Remark 5.3** We do not consider the case where the initial data  $u_0$  is not radially symmetric (see, e.g., [16, 17, 18, 31] for the case  $a = 0$ ).

We first prove a simple lemma.

**Lemma 11.** *Let  $T \in (0, \infty)$ , and assume that a function  $F : [0, T) \rightarrow (0, \infty)$  is continuous, and  $\lim_{t \rightarrow T} F(t) = \infty$ . Then, there exists a sequence  $\{t_k\}$  such that  $t_k \rightarrow T$  and*

$$\lim_{k \rightarrow \infty} \frac{\int_0^{t_k} F(\tau) d\tau}{F(t_k)} = 0. \quad (5.3)$$

*Proof.* We follow the argument in the proof of Lemma 2.1 of [4]. We put

$$G(t) = \int_0^t F(\tau) d\tau, \quad t \in [0, T).$$

If  $\lim_{t \rightarrow T} G(t) < \infty$ , then (5.3) holds for any sequence  $\{t_k\}$  satisfying  $t_k \rightarrow T$ . So, we assume that  $\lim_{t \rightarrow T} G(t) = \infty$ . Then, since  $\lim_{t \rightarrow T} \log G(t) = \infty$ , we see that

$$\limsup_{t \rightarrow T} \frac{F(t)}{G(t)} = \limsup_{t \rightarrow T} \frac{d}{dt} \log G(t) = \infty,$$

which shows that there exists a sequence  $\{t_k\}$  such that  $t_k \rightarrow T$  and (5.3).  $\square$

The following variational characterization of the ground state  $Q$  plays an important role in the proof of Theorem 10.

**Lemma 12.** *Let  $p = 1 + 4/n$ , and  $Q$  be the ground state of (5.1). Then we have*

$$\|Q\|_{L^2} = \inf\{\|v\|_{L^2} : v \in H^1(\mathbb{R}^n) \setminus \{0\}, E(v) \leq 0\}.$$

For the proof of Lemma 12, see Weinstein [29] and also [18].

*Proof of Theorem 10.* We follow the argument in the proof of Theorem 6.6.7 of [2]. We put  $\sigma = 2 + 4/n$ . We denote by  $H_{rad}^1(\mathbb{R}^n)$  the set of radially symmetric functions in  $H^1(\mathbb{R}^n)$ . Since the initial data  $u_0 \in H_{rad}^1(\mathbb{R}^n)$ , we see that  $u(t) \in H_{rad}^1(\mathbb{R}^n)$  for all  $t \in [0, T)$ . By the energy identity (1.6), we have

$$E(u(t)) = E(u_0) - a \int_0^t K(u(\tau)) d\tau, \quad t \in [0, T). \quad (5.4)$$

By (1.7), the Gagliardo-Nirenberg inequality (1.3) and (1.5), we have

$$\begin{aligned} |K(u(t))| &\leq \|\nabla u(t)\|_{L^2}^2 + \|u(t)\|_{L^\sigma}^\sigma \\ &\leq \|\nabla u(t)\|_{L^2}^2 + C\|u(t)\|_{L^2}^{\sigma-2} \|\nabla u(t)\|_{L^2}^2 \\ &\leq (1 + C\|u_0\|_{L^2}^{\sigma-2}) \|\nabla u(t)\|_{L^2}^2 \end{aligned}$$

for all  $t \in [0, T)$ . Moreover, by Remark 5.1, we have  $\lim_{t \rightarrow T} \|\nabla u(t)\|_{L^2}^2 = \infty$ . Thus, by Lemma 11, there exists a sequence  $\{t_k\}$  such that  $t_k \rightarrow T$  and

$$\lim_{k \rightarrow \infty} \frac{\int_0^{t_k} K(u(\tau)) d\tau}{\|\nabla u(t_k)\|_{L^2}^2} = 0. \quad (5.5)$$

Here, we put

$$\lambda_k = 1/\|\nabla u(t_k)\|_{L^2}, \quad v_k(x) = \lambda_k^{n/2} u(t_k, \lambda_k x).$$

Then, we see that  $v_k \in H_{rad}^1(\mathbb{R}^n)$  and

$$\|v_k\|_{L^2} = \|u(t_k)\|_{L^2} \leq \|u_0\|_{L^2}, \quad \|\nabla v_k\|_{L^2} = 1. \quad (5.6)$$

By (5.4) and (5.5), we have

$$E(v_k) = \frac{1}{2} - \frac{1}{\sigma} \|v_k\|_{L^\sigma}^\sigma = \lambda_k^2 E(u_0) - a \lambda_k^2 \int_0^{t_k} K(u(\tau)) d\tau \rightarrow 0 \quad (5.7)$$

as  $k \rightarrow \infty$ . Since  $\{v_k\}$  is bounded in  $H^1(\mathbb{R}^n)$  by (5.6), there exists a subsequence of  $\{v_k\}$  (we still denote it by the same letter) and  $w \in H^1(\mathbb{R}^n)$  such that

$$v_k \rightharpoonup w \text{ weakly in } H^1(\mathbb{R}^n). \quad (5.8)$$

Since  $\{v_k\} \subset H_{rad}^1(\mathbb{R}^n)$  and  $n \geq 2$ , it follows from (5.8) and the compactness of the embedding  $H_{rad}^1(\mathbb{R}^n) \hookrightarrow L^\sigma(\mathbb{R}^n)$  (see Strauss [21]) that

$$v_k \rightarrow w \text{ strongly in } L^\sigma(\mathbb{R}^n). \quad (5.9)$$

By (5.7), (5.8) and (5.9), we have  $\|w\|_{L^\sigma}^\sigma = \sigma/2$ , especially  $w \neq 0$ , and

$$E(w) \leq \liminf_{k \rightarrow \infty} E(v_k) = 0.$$

Therefore, by Lemma 12, we have

$$\|Q\|_{L^2} \leq \|w\|_{L^2}. \quad (5.10)$$

Moreover, by (5.8) and since  $\rho(t_k) \rightarrow \infty$ , for any  $M > 0$ , we have

$$\begin{aligned} \|w\|_{L^2(\{|x| < M\})} &= \lim_{k \rightarrow \infty} \|v_k\|_{L^2(\{|x| < M\})} = \lim_{k \rightarrow \infty} \|u(t_k)\|_{L^2(\{|x| < M\lambda_k\})} \\ &\leq \liminf_{k \rightarrow \infty} \|u(t_k)\|_{L^2(\{|x| < \rho(t_k)\lambda_k\})}. \end{aligned}$$

Since  $M > 0$  is arbitrary, by (5.10), we have

$$\|Q\|_{L^2} \leq \|w\|_{L^2} \leq \liminf_{k \rightarrow \infty} \|u(t_k)\|_{L^2(\{|x| < \rho(t_k)\lambda_k\})},$$

which shows (5.2).  $\square$

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