ASYMPTOTIC BEHAVIOR AND REGULARITY FOR NONLINEAR DISSIPATIVE WAVE EQUATIONS IN $\mathbb{R}^n$

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We discuss several results on asymptotic behavior and regularity for wave equations with nonlinear damping $u_{tt} - \Delta u + |u_t|^{m-1}u_t = 0$ in $\mathbb{R}_+ \times \mathbb{R}^n$, where $m > 1$. In particular, we show that $\nabla u \in L^{m+1}(I \times \mathbb{R}^n)$ for all initial data $(u, u_t)|_{t=0} \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ when $m \leq (n+2)/n$ and $I \subset \mathbb{R}_+$ is a compact interval. We also consider global well-posedness in Sobolev spaces $H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)$ for $k > 2$. The strength of nonlinear damping is critical when $m = n/(n-2)$ and $n \geq 3$, which makes classical techniques ineffective. We outline the proof of global well-posedness for $m = 3$ and $n = 3$ (critical) under the additional condition that $(u, u_t)|_{t=0}$ are spherically symmetric. Finally, we present scattering results for spherically symmetric global solutions.

Keywords: wave equation, nonlinear damping, regularity, asymptotic behavior.

1. Introduction

The asymptotic behavior and regularity of solutions to wave equations with nonlinear damping ($m > 1$)

$$u_{tt} - \Delta u + |u_t|^{m-1}u_t = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{1}$$

are not only challenging questions but also starting points for understanding basic mechanisms in nonlinear hyperbolic equations, such as scattering-damping[18–20] and smoothing-focussing.[3,9,14] A classical result of Lions and Strauss[15] states that (1) is globally well-posed for all initial data

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n, \tag{2}$$

with $(u_0, u_1) \in H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)$ and $1 \le k \le 2$; see also Joly, Metivier and Rauch,[9] Liang,[14] and Serrin, Todorova and Vitillaro.[22] Moreover, if

$$u_0 \in H^2(\mathbb{R}^n), \quad u_1 \in H^1(\mathbb{R}^n) \cap L^{2m}(\mathbb{R}^n),$$

problem (1), (2) has a unique global solution u with the following properties:

(a) $u \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}^n)), \quad u_t \in L^\infty(\mathbb{R}_+, H^1(\mathbb{R}^n))$;

(b) $E_1(u,t) = E_1(u,t_0) - \int_{t_0}^t \int |u_s|^{m+1} dxds, \quad t \ge t_0,$

where $E_k(u,t) = \dfrac{1}{2} \sum_{|\alpha|=k-1} \int (|\nabla^\alpha u_t|^2 + |\nabla^\alpha \nabla u|^2) dx;$

(c) $E_1(u,t) \le E_1(u,t_0), \quad E_2(u,t) \le E_2(u,t_0), \quad t \ge t_0;$

(d) $\mathrm{supp}(u, u_t) \subset \{|x| \le t + R\}$ if $\mathrm{supp}(u_0, u_1) \subset \{|x| \le R\}$.

Hence, $E_1(u,t)$ and $E_2(u,t)$ are non-increasing functions of t. A natural question is whether these norms decay to zero or not as $t \to \infty$. The answers are known in several cases described in Section 2. Another interesting problem is suggested by identity (b) which implies $u_t \in L^{m+1}(\mathbb{R}_+ \times \mathbb{R}^n)$ even for data $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Section 3 shows that the gain of regularity in ∇u is similar if $m \le (n+2)/n$: $\nabla u \in L^{m+1}(I \times \mathbb{R}^n)$ for every compact $I \subset \mathbb{R}_+$. The final Section 4 deals with the question $E_k(u,t) < \infty$ at all $t \in \mathbb{R}_+$ for $k > 2$ and arbitrarily large $E_k(u,0)$. It is expected, based on the invariant scaling of (1), that either $m < \infty$ and $n = 1, 2$, or $m \le n/(n-2)$ and $n \ge 3$ will be a sufficient condition. However, the critical cases $m = n/(n-2)$, $n \ge 3$ remain open as they are difficult to study by current perturbation techniques. This paper outlines authors contribution[25] which covers $m = 3$, $n = 3$, and spherically symmetric data in $H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$, $k \ge 3$.

2. Asymptotic behavior of energy

The long time behavior of solutions to the Cauchy problem (1), (2) is not well-understood yet. We should mention that the boundary value problem $x \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, is quite different. Nakao[21] and Haraux[7] have found polynomial decay rates of $E_1(u,t)$ under the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. The current state of this problem and its generalizations for localized damping and source is presented in Lasiecka and Toundykov.[13] A surprising similarity with the Cauchy problem is that no decay estimates are known for $E_2(u,t)$ and higher-order norms.

There are three types of results about the asymptotic behavior of (1), (2) with data $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$: decay estimates for $E_1(u,t)$, lower bounds on $E_1(u,t)$, and $E_1(u-u_+,t) \to 0$ as $t \to \infty$ for a finite-energy solution u_+ of the wave equation, i.e., scattering in the energy space. Clearly, the existence of scattering states implies that the energy approaches a non-zero constant. Below is a short review of significant results.

A sufficient condition for scattering, due to Motai and Mochizuki,[20] is $m > 1 + 2/(n-1)$, $n \geq 2$. In such cases, the energy $E_1(u,t)$ can not decay as $t \to \infty$. The result is generalized by Matsuyama[17] to certain exterior domains in $n = 3$. Both papers contain direct proofs that $E_1(u,t)$ does not decay, which avoid the existence of scattering states u_+.

The strongest decay estimate for problem (1), (2) has long been logarithmic. For $1 < m < 1 + 2/n$ and $0 < \mu < 2/(m-1)$, Mochizuki and Motai[19] have established

$$E_1(u,t) \leq C_\mu \{\ln(2+t)\}^{-\mu}, \tag{3}$$

where C_μ depends on μ and (u_0, u_1). The gap between conditions for scattering and energy decay means that the asymptotic behavior of $E_1(u,t)$ is unknown if $1 + 2/n \leq m \leq 1 + 2/(n-1)$. Estimate (3) is very weak but it may be sharp in the critical cases $n \geq 2$, $m = 1 + 2/(n-1)$. Kubo[12] has found such modified asymptotic profiles in $n = 2$, $m = 3$.

The logarithmic decay rate of energy is unlikely to be sharp, however, if $1 < m < 1 + 2/n$. Then the invariant scaling and nonlinear diffusion phenomenon for (1) strongly suggest polynomial rates depending on n and m. To explain the first argument, we consider a change of variables

$$u(t,x) \mapsto u_\lambda(t,x) = \lambda^{(2-m)/(m-1)} u(\lambda(t-t_0), \lambda(x-x_0)), \tag{4}$$

with $\lambda > 0$ and $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Equation (1) is invariant under these transformations. Since $\{u_\lambda(t,x)\}$ resembles a delta family concentrating at (t_0, x_0), as $\lambda \to \infty$, it can be used for testing conjectures about the asymptotic behavior and regularity of u. Changes may occur at critical m where the energy is λ-invariant. We readily calculate

$$E_k(u_\lambda, t) = \lambda^{2/(m-1)+2(k-1)-n} E_k(u, \lambda(t-t_0)) \tag{5}$$

and set $t = 1$, $t_0 = 0$, and $k = 1$ to obtain

$$E_1(u,\lambda) = \lambda^{n-2/(m-1)} E_1(u_\lambda, 1).$$

Thus, the behavior of $u(\lambda, x)$ as $\lambda \to \infty$ is determined by the relative smoothness of $u_\lambda(1,x)$ compared to $u_\lambda(0,x)$; any non-trivial smoothing estimate of $u_\lambda(1,x)$ will imply a decay estimate of $u(\lambda,x)$. The former type of results is very plausible since the nonlinear damping has a regularizing effect on solutions. Notice that always $E_1(u_\lambda, 1) \to \infty$ as $\lambda \to \infty$, so the energy of u is not expected to decay when $m \geq 1 + 2/n$.

To present the second argument in favor of polynomial decay rates, we assume that (1) exhibits the nonlinear diffusion phenomenon. Here the relevant correspondence is $u_t \approx v$, where v solves the fast diffusion equation

$$(v^m)_t - \Delta v = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Every positive solution v has a polynomial asymptotic profile

$$v(t, x) \sim t^{-n\gamma}(\alpha + \beta|x|^2/t^{2m\gamma})^{-1/(m-1)}, \tag{6}$$

with α, $\beta > 0$, and $\gamma = (n - m(n-2))^{-1}$; see Carrillo and Vázquez[4] and the references therein. The wave equation with a nonlinear damping (1) is formally transformed into the fast diffusion equation if u_{tt} is neglected, the remaining terms are differentiated with respect to t, and u_t is replaced by v. Such manipulations can not be justified, although they are valid when the damping is linear; see Matsumura.[16]

Theorem 2.1. *Let u be a solution of (1), (2) with compactly supported $(u_0, u_1) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. If $n \geq 3$ and $1 < m \leq (n+2)/(n+1)$, then*

$$E_1(u, t) \leq C_a t^{-a}, \quad t \to \infty,$$

where $a > 0$ depends only on m and n, while C_a depends on a and (u_0, u_1).

Unfortunately, the above decay rate[26] is implicit. Here the the lack of control on $\|u\|_{L^2}$ is the main difficulty. This is also an essential difference with the wave equation in a bounded domain or the Klein-Gordon equation $u_{tt} - \Delta u + u + |u_t|^{m-1}u_t = 0$. The proof of Theorem 2.1 uses "parabolic" effects coming from the presence of damping in (1). A crucial idea is to introduce weights similar to the asymptotic profile of fast diffusion (6).

3. Smoothing effects

The regularizing effect of nonlinear damping on solutions of (1), (2) has already been studied. Examples are constructed of piecewise C^2 data with jump discontinuities which are partially smoothed during the evolution when $m > 1 + 2/(n-1)$; see Joly, Metivier, and Rauch[9] (radial data) and Liang[14] (general data).

We consider the complementary range $1 < m \leq 1 + 2/n$ and general data $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Recall that $u_t \in L^{m+1}(\mathbb{R}_+ \times \mathbb{R}^n)$, so u_t is more regular than u_1 for almost all $t > 0$; see Sec. 1, identity (b).

The partial smoothing of u_t is a significant difference with the linear wave equation. A natural question is whether the regularizing effect extends to ∇u. The answer is affirmative, if the nonlinear damping is relatively weak.

Theorem 3.1. *Assume that $1 < m \leq 1 + 2/n$. For every compact interval $I \subset \mathbb{R}_+$, the solution u of (1) satisfies*

$$\|\nabla u\|_{L^{m+1}(I\times R^n)} \leq C \left(E_1^{1/(m+1)}(u,0) + E_1^{m/(m+1)}(u,0) \right),$$

where C depends only on m, n, and I.

We begin with a simple estimate for the wave operator $\Box = \partial_t^2 - \Delta$.

Lemma 3.1. *Assume that $1 < p \leq q < \infty$ and $1/p - 1/q = 1/(n+1)$. There exists C depending on p, q, and n, such that*

$$\|\nabla u\|_{L^q(R^{n+1})} \leq C\|\partial_t u\|_{L^q(R^{n+1})} + C\|\Box u\|_{L^p(R^{n+1})}$$

for every compactly supported distribution $u \in \mathcal{E}'(\mathbb{R}^{n+1})$ which has finite norms on the right side.

The proof of Lemma 3.2 is postponed for the end of this section.

Proof of Theorem 3.1. Let $I \subset J$ be compact intervals in $\mathbb{R}_+$ and $\chi \in C_0^\infty(\mathbb{R}_+)$ be a cut-off function satisfying

$$\chi(t) = \begin{cases} 1, & \text{if } \ t \in I, \\ 0, & \text{if } \ t \in \mathbb{R} \setminus J. \end{cases}$$

We denote the first two derivatives of χ by χ' and χ''. Applying Lemma 3.1 to χu, where u is a solution of (1), we have

$$\|\nabla_x(\chi u)\|_{L^q(R^{n+1})} \leq C\|\partial_t(\chi u)\|_{L^q(R^{n+1})} + C\|\Box(\chi u)\|_{L^p(R^{n+1})}.$$

Since

$$\Box(\chi u) = -\chi|u_t|^{m-1}u_t + 2\chi' u_t + \chi'' u, \quad \partial_t(\chi u) = \chi u_t + \chi' u,$$

the initial estimate becomes

$$\begin{aligned} \|\chi\nabla_x u\|_{L^q(R^{n+1})} \leq\ & C(\|\chi u_t\|_{L^q(R^{n+1})} + \|\chi' u\|_{L^q(R^{n+1})}) \\ & + C\|\chi|u_t|^m\|_{L^p(R^{n+1})} \\ & + C(\|\chi' u_t\|_{L^p(R^{n+1})} + \|\chi'' u\|_{L^p(R^{n+1})}). \end{aligned} \tag{7}$$

The strongest regularization occurs when

$$q = m+1, \qquad p = (m+1)(n+1)/(m+n+2), \tag{8}$$

as the nonlinear damping requires $q \leq m+1$ for $u_t \in L^q(\mathbb{R}_+ \times \mathbb{R}^n)$.

Notice also that $m \leq 1 + 2/n$ implies $p \leq (m+1)/m$. Hence, the most singular term in (7) is estimated by Hölder's inequality:

$$\|\chi|u_t|^m\|_{L^p(R^{n+1})} \leq C\|u_t\|^m_{L^{m+1}(J\times R^n)} \leq CE_1^{m/(m+1)}(u,0). \tag{9}$$

We can assume that $u(t,\cdot)$ is compactly supported, since the estimates are local in time and the equation has a finite speed of propagation.

The other terms in (7) are trivially bounded by $E_1(u,0)$. Recall that the integration takes place on a compact set $\{(t,x) : t \in J,\ |x| \leq t+R\}$ and $p \leq (m+1)/m < 2$, where p and q are defined in (8).

We begin with the norms involving u_t on the right side of (7):

$$\begin{aligned} &\|\chi u_t\|_{L^q(R^{n+1})} + \|\chi' u_t\|_{L^p(R^{n+1})} \\ &\leq \|u_t\|_{L^{m+1}(J\times R^n)} + C\|u_t\|_{L^2(J\times R^n)} \\ &\leq CE_1^{1/(m+1)}(u,0) + CE_1^{1/2}(u,0). \end{aligned} \tag{10}$$

The Sobolev embedding $H^1(\mathbb{R}^{n+1}) \hookrightarrow L^q(\mathbb{R}^{n+1})$ for $q \leq 2+2/n$ implies

$$\begin{aligned} \|\chi' u\|_{L^q(R^{n+1})} &\leq C(\|u_t\|_{L^2(J\times R^n)} + \|\nabla u\|_{L^2(J\times R^n)}) \\ &\leq CE_1^{1/2}(u,0). \end{aligned} \tag{11}$$

Finally, we use $H^1(\mathbb{R}^{n+1}) \hookrightarrow L^2(\mathbb{R}^{n+1})$ and $p < 2$:

$$\begin{aligned} \|\chi'' u\|_{L^p(R^{n+1})} &\leq \|\nabla u\|_{L^2(J\times R^n)} \\ &\leq CE_1^{1/2}(u,0). \end{aligned} \tag{12}$$

Substituting estimates (9), (10), (11), and (12) into (7), we obtain

$$\|\nabla_x u\|_{L^{m+1}(I\times R^n)} \leq C\left(E_1^{1/2}(u,0) + E_1^{1/(m+1)}(u,0) + E_1^{m/(m+1)}(u,0)\right).$$

Since $1/(m+1) \leq 1/2 \leq m/(m+1)$, we can drop $E_1^{1/2}(u,0)$ from the last estimate. The proof of Theorem 3.1 is complete.

Proof of Lemma 3.1. Define the Fourier transform $\mathcal{F}$ as

$$\mathcal{F}u(\tau,\xi) = \int_{R^{n+1}} e^{-i(\tau t + \xi x)} u(t,x)\,dx\,dt, \quad (\tau,\xi) \in \mathbb{R}^{n+1},$$

and denote its inverse by $\mathcal{F}^{-1}$. From $\mathcal{F}(\partial_t u) = i\tau\mathcal{F}u$ and $\mathcal{F}(\nabla u) = i\xi\mathcal{F}u$,

$$\mathcal{F}(\Box u) = (-\tau^2 + |\xi|^2)\mathcal{F}u.$$

To express ∇u in terms of $\partial_t u$ and $\Box u$, we rewrite this identity as

$$|\xi|\mathcal{F}u = |\tau|\mathcal{F}u + (|\xi| + |\tau|)^{-1}\mathcal{F}(\Box u)$$

and apply $\mathcal{F}^{-1}$ to both sides:

$$\begin{aligned}\nabla u = \mathcal{F}^{-1}&\left(\xi|\xi|^{-1}\mathrm{sign}(\tau)\mathcal{F}(\partial_t u)\right) \\ &+ i\mathcal{F}^{-1}\left(\xi|\xi|^{-1}(|\xi|+|\tau|)^{-1}\mathcal{F}(\Box u)\right).\end{aligned} \tag{13}$$

Here $\mathrm{sign}(\tau) = \tau/|\tau|$ for $\tau \neq 0$. It is well known[5] that $\xi/|\xi|$ and $\mathrm{sign}(\tau)$ are $L^q(\mathbb{R}^{n+1})$ Fourier multipliers, while $(|\xi|+|\tau|)^{-1}$ is an $L^p(\mathbb{R}^{n+1}) - L^q(\mathbb{R}^{n+1})$ Fourier multiplier if $1/p - 1/q = 1/(n+1)$ and $1 < q < \infty$. Using (13),

$$\begin{aligned}\|\nabla u\|_{L^q(R^{n+1})} &\leq \|\mathcal{F}^{-1}(\xi|\xi|^{-1}\mathrm{sign}(\tau)\mathcal{F}(\partial_t u))\|_{L^q(R^{n+1})} \\ &\quad + \|\mathcal{F}^{-1}(\xi|\xi|^{-1}(|\xi|+|\tau|)^{-1}\mathcal{F}(\Box u))\|_{L^q(R^{n+1})} \\ &\leq C\|\partial_t u\|_{L^q(R^{n+1})} + C\|\Box u\|_{L^p(R^{n+1})}.\end{aligned}$$

This completes the proof of Lemma 3.1.

4. Regularity and scattering for critical damping

The global well-posedness of (1) is a highly nontrivial problem in Sobolev spaces $H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)$ with $k > 2$ since the monotonicity of $|u_t|^{m-1}u_t$ no longer implies *a priori* estimates. Standard techniques treat the nonlinear damping as a perturbation and restrict its growth $m \leq n/(n-2)$ if $n \geq 3$; this is implied by scaling laws (4), (5) and $E_2(u,t) \leq E_2(u,0)$:

$$E_2(u_\lambda, t) = \lambda^{2/(m-1)+2-n}E_2(u, \lambda t) \leq E_2(u, 0), \quad \lambda \to \infty,$$

for $m \geq n/(n-2)$, $n \geq 3$. Hence, the existence of concentrating solutions (increasing norms of order $k > 2$) can not be excluded for large m. There is some possibility to handle the critical case $m = n/(n-2)$ by improved classical techniques, similarly to the nonlinear wave and Schrodinger equations; see Grillakis,[6] Shatah and Struwe,[23] Bahouri and Gérard[1] for the former and Bourgain,[2] Tao, Visan, and Zhang[24] for the latter. So far the global well-posedness of (1) is established along these lines in a special case: $m = 3$, $n = 3$, and radial data.[25] We will present the main ideas and analytic tools.

Our result concern mainly large data, as the global existence and asymptotic behavior are well-understood for small data; see Klainerman and Ponce,[11] Hörmander,[8] and the recent work of Matsuyama.[17]

Let us define the Sobolev spaces of spherically symmetric functions

$$H^k_{\mathrm{rad}}(\mathbb{R}^3) = \{u \in H^k(\mathbb{R}^3) : u \text{ is a function of } |x|\},$$

for $k \geq 1$. These are invariant under the evolution determined by (1), (2).

We write $Du = (\nabla u, u_t)$ for the space-time derivative of u. The following is partial answer to the question about global well-posedness.[25]

Theorem 4.1. *Let $m = 3$, $n = 3$, and $(u_0, u_1) \in H^3_{\mathrm{rad}}(\mathbb{R}^3) \times H^2_{\mathrm{rad}}(\mathbb{R}^3)$. Then problem (1), (2) admits a unique global solution u, such that*

$$D^\alpha u \in C(\mathbb{R}_+, H^{3-|\alpha|}_{\mathrm{rad}}(\mathbb{R}^3)), \qquad |\alpha| \leq 3.$$

Moreover, u satisfies the estimates

$$\sum_{|\alpha|\leq 2} \int_0^t \|D^\alpha u(s)\|^2_{L^\infty}\, ds \leq C_1(u_0, u_1), \quad t \geq 0,$$

$$\sum_{|\alpha|\leq 2} E_1(D^\alpha u, t) \leq C_2(u_0, u_1), \quad t \geq 0,$$

for implicit constants $C_k(u_0, u_1)$, $k = 1, 2$, which are finite whenever the norm $\|u_0\|_{H^3} + \|u_1\|_{H^2}$ is finite.

Remark 4.1. We rely on the "forbidden" $L^1_t L^2_x - L^2_t L^\infty_x$ Strichartz estimate for the wave equation in $\mathbb{R} \times \mathbb{R}^3$; see Klainerman and Machedon.[10] The estimate is valid only for spherically symmetric solutions which explains the condition for spherically symmetric data.

Proof of Theorem 4.1 (outline). Recall a useful fact: for $|\alpha| \leq 1$,

$$E_1(D^\alpha u, t) + (2|\alpha| + 1) \int_0^t \|u_s D^\alpha u_s\|^2_{L^2}\, ds = E_1(D^\alpha u, 0).$$

Step 1. Let u be a local solution of (1), (2), such that $E_3(u, t) < \infty$ for $t \in [0, T_*)$. The equation for $D^\alpha u$, $|\alpha| = 2$, involves non-dissipative terms:

$$\Box D^\alpha u + 3u_t^2 D^\alpha u_t + \sum_{\beta+\gamma=\alpha} c_{\beta,\gamma} u_t D^\beta u_t D^\gamma u_t = 0, \tag{14}$$

where $c_{\beta,\gamma}$ are constants. From $E_2(u, t) \leq E_2(u, 0)$, these are bounded by

$$\int_0^t \|D^\alpha u(s)\|^2_{L^\infty}\, ds, \quad |\alpha| \leq 2. \tag{15}$$

Step 2. We can handle $L^2_t L^\infty_x$ norms by means of

$$\left(\int_0^t \|u(s)\|^2_{L^\infty}\, ds\right)^{1/2} \leq C E_1^{1/2}(u, 0) + C \int_0^t \|\Box u(s)\|_{L^2}\, ds,$$

which is an endpoint Strichartz estimate. If $|\alpha| = 1$, the inequality leads to

$$\left(\int_0^t \|D^\alpha u(s)\|^2_{L^\infty}\, ds\right)^{1/2} \leq C E_2^{1/2}(u, 0) + C \int_0^t \|u_s^2(s) D^\alpha u_s(s)\|_{L^2}\, ds.$$

It is not trivial to bound the right side by the $L^2_t L^\infty_x$ norm on the left side. We need a simple "non-concentration" proposition.

Prop 4.1. Assume that $(u_0, u_1) \in H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ and $|\alpha| = 1$. Let u be the solution of problem (1), (2) extended on a maximal interval $[0, T_*)$. For every $\epsilon > 0$ there exists $\delta \in (0, T_0)$, such that

$$\sup_{t\in[T_0,T_*)} \int_{t-\delta}^{t} (\|u_s(s)\|_{L^4}^4 + \|u_s(s)D^\alpha u_s(s)\|_{L^2}^2)\, ds < \epsilon.$$

The result follows from $\|u_s(s)\|_{L^4}^4 + \|u_s(s)D^\alpha u_s(s)\|_{L^2}^2 \in L^1([0, T_*))$.

Applying Prop. 4.1, we find $C > 0$ and $\delta = \delta(u_0, u_1, \alpha) > 0$, such that

$$\left(\int_0^t \|D^\alpha u(s)\|_{L^\infty}^2\, ds\right)^{1/2} \le CE_2^{1/2}(u, 0)(1 + CE_2^{1/2}(u, 0))^{1+t/\delta}, \quad (16)$$

for all $t \in [0, T_*)$. The estimate for $|\alpha| = 2$ in (15) is similar.

Step 3. To complete the proof of $E_3(u, t) < \infty$, we combine the standard energy estimate from (14) and the $L_t^2 L_x^\infty$-estimate (16) with $|\alpha| = 1, 2$.

The solutions constructed in Theorem 4.1 are asymptotically free, since the nonlinear term u_t^3 is supercritical for scattering theory in $\mathbb{R}^3$.

Theorem 4.2. *Let $m = 3$, $n = 3$, and $(u_0, u_1) \in H^3_{\rm rad}(\mathbb{R}^3) \times H^2_{\rm rad}(\mathbb{R}^3)$. The global solution u of problem (1), (2) is asymptotically free:*

$$\|D^\alpha u(t) - D^\alpha u_+(t)\|_2 \to 0, \qquad t \to \infty,$$

for $1 \le |\alpha| \le 3$, where u_+ is a solution of $\Box u_+ = 0$ with initial data

$$Du_+(x, 0) \in H^2_{\rm rad}(\mathbb{R}^3) \times H^1_{\rm rad}(\mathbb{R}^3).$$

Proof of Theorem 4.2 (outline). We use the estimates in Theorem 4.1 and Hölder's inequality to show a sufficient condition for scattering:

$$\int_0^\infty \|D^\alpha u_t^3(t)\|_2\, dt < \infty, \quad 0 \le |\alpha| \le 2.$$

References

1. Bahouri, H.; Gérard, P., *High frequency approximation of solutions to critical nonlinear wave equations*, Amer. J. Math. 121 (1999), 131–175.
2. Bourgain, J., *Global wellposedness of defocusing critical nonlinear Schrdinger equation in the radial case*, J. Amer. Math. Soc. 12 (1999), no. 1, 145–171.
3. Carles, R. and Rauch, J., *Focusing of Spherical Nonlinear Pulses in* $\mathbb{R}^{1+3}$, Proc. Amer. Math. Soc. 130 (2002), 791–804.
4. Carrillo, Jos A.; Vázquez, Juan L., *Fine asymptotics for fast diffusion equations*, Comm. Partial Differential Equations 28 (2003), no. 5-6, 1023–1056.
5. Duoandikoetxea, Javier, *Fourier analysis*, translated and revised by David Cruz-Uribe, Providence, RI: American Mathematical Society, (2000).

6. Grillakis, Manoussos, *Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity*, Ann. of Math. (2) 132 (1990), no. 3, 485–509.
7. Haraux, Alain, *Comportement l'infini pour une quation d'ondes non linaire dissipative*, C. R. Acad. Sci. Paris Sr. A-B 287 (1978), no. 7, A507–A509.
8. Hörmander, Lars, *Lectures on Nonlinear Wave Equations*, Mathématiques & Applications, 26, Springer Verlag, Berlin, 1997.
9. Joly, J.-L., Métivier, G., and Rauch, J., *Nonlinear hyperbolic smoothing at a focal point*, Michigan Math. J., 47 (2000), 285–312.
10. Klainerman, S.; Machedon, M., *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. 46 (1993), no. 9, 1221–1268.
11. Klainerman, S.; Ponce, G, *Global, small amplitude solutions to nonlinear evolution equations*, Comm. Pure Appl. Math., 63, pp. 133–141, 1983.
12. Kubo, Hideo, *Large-time behavior of solutions to semiliniear wave equations with dissipative terms*, preprint 2006.
13. Lasiecka, Irena; Toundykov, Daniel, *Energy decay rates for the semilinear wave equation with nonlinear localized damping and a source terms - an intrinsic approach*, preprint (2005).
14. Liang, J., *Nonlinear hyperbolic smoothing at a focal point*, preprint (2004).
15. Lions, Jacques-Louis; Strauss, W. A., *Some non-linear evolution equations*, Bull. Soc. Math. France 93 (1965), 43–96.
16. Matsumura, Akitaka, *On the asymptotic behavior of solutions of semi-linear wave equations*, Publ. Res. Inst. Math. Sci. 12 (1976/77), no. 1, 169–189.
17. Matsuyama, T., *Asymptotics for the nonlinear dissipative wave equation*, Transactions AMS 355 (2003), 865–899.
18. Mochizuki, K., *Decay and asymptotics for wave equations with dissipative terms*, Lecture Notes in Phys. 39 (1975), Springer-Verlag, 486–490.
19. Mochizuki, Kiyoshi; Motai, Takahiro, *On energy decay-nondecay problems for wave equations with nonlinear dissipative term in* R^N, J. Math. Soc. Japan 47 (1995), no. 3, 405–421.
20. Motai, T. and Mochizuki, K., *On asymptotic behaviors for wave equation with a nonlinear dissipative term in* $\mathbb{R}^N$, Hokkaido Math. J. 25 (1996), 119–135.
21. Nakao, Mitsuhiro, *Asymptotic stability of the bounded or almost periodic solution of the wave equation with nonlinear dissipative term*, J. Math. Anal. Appl. 58 (1977), no. 2, 336–343.
22. Serrin, J.; Todorova, G.; Vitillaro, E., *Existence for a nonlinear wave equation with damping and source terms*, Diff. Integ. Equations 16 (2003), no. 1, 13–50.
23. Shatah, Jalal; Struwe, Michael, *Regularity results for nonlinear wave equations*, Ann. of Math. (2) 138 (1993), no. 3, 503–518.
24. Tao, T.; Visan, M.; Zhang, X., *Global well-posedness and scattering for the mass-critical nonlinear Schrödinger equation for radial data in high dimensions*, preprint 2006, to appear in Duke Math J.
25. Todorova, G.; Uğurlu, D.; Yordanov, B., *Regularity and Scattering for the Wave Equation with a Critical Nonlinear Damping*, preprint 2007.
26. Todorova, G.; Yordanov, B., *The energy decay problem for wave equations with nonlinear dissipative terms in* $\mathbb{R}^n$, Indiana University Math. J. 56 (2007), no. 1, 389–416.