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Critical exponent for a nonlinear wave equation with damping

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Abstract. In this study we solve the critical exponent problem for the nonlinear wave equation with damping. We show that this critical exponent coincides with the famous Fujita critical exponent for the heat equation. We also observe a futher interesting phenomenon due to the presence of the damping term. The "real" support of the solution is strongly suppressed by the damping and is concentrated in the ball $|x| < t^{1/2}$, which is much smaller then $|x| < t + K$. Outside of the ball $\mathrm{B}(t^{1/2+\delta})$ with $\delta > 0$ the solution decays exponentially. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Exposant critique pour l'équation des ondes avec un terme dissipatif

Résumé. *Dans cette étude nous résolvons le problème concernant l'exposant critique pour l'équation des ondes avec un terme dissipatif. On démontre que cet exposant critique coïncide avec le célèbre exposant critique de Fujita. On observe un autre phénomène intéressant, dû à la présence du terme dissipatif. Notamment, le support « réel » de la solution est fortement compressé par le terme dissipatif. Plus précisément, on démontre qu'en-dehors du domaine* $\mathrm{B}(t^{1/2+\delta})$, *oú* $\delta > 0$, *la solution décroît d'une manière exponentielle.* © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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On considère le problème de Cauchy pour l'équation des ondes non linéaire avec un terme dissipatif :

$$\Box u + u_t = |u|^p \quad \text{pour } (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1}$$

$$u(0,x) = \varepsilon u_0, \quad u_t(0,x) = \varepsilon u_1, \tag{2}$$

où $\varepsilon > 0$ et où les données initiales ayant un support compact appartiennent à l'espace énergétique.

Notre analyse sera centrée sur l'exposant critique $p_{\mathrm{c}}(n)$, lequel est défini comme étant un nombre possédant les caractéristiques suivantes :

– lorsque $p_{\mathrm{c}}(n) < p$, toutes les solutions de (1), (2) avec des données initiales suffisamment petites, sont globales, tandis que lorsque $1 < p < p_{\mathrm{c}}(n)$, toutes les solutions de (1), (2) avec des données moyennes positives explosent dans un temps fini, quelque petites que puissent être ces données.

Note présentée par Jacques-Louis LIONS.

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Il est bien connu qu'en l'absence de terme dissipatif l'exposant critique de l'équation des ondes non linéaire $\Box u = |u|^p$, est la racine positive $p_w(n)$ du polynôme $(n-1)p^2 - (n+1)p - 2$, où n est la dimension de l'espace. La preuve de cette circonstance, connue sous le nom de conjecture de Strauss, a requis presque vingt ans d'efforts de nombreux mathématiciens, à commencer par Glassey [3], John [4], Sideris [7], Zhou [10], pour finir avec Lindblad et Sogge [6], Georgiev, Lindblad et Sogge [2], et Tataru [9].

Dans cette étude nous résolvons le problème concernant l'exposant critique de l'équation des ondes non linéaire avec un terme dissipatif. Notre résultat principal consiste dans le fait que l'exposant critique $p_\mathrm{c}(n)$ de l'équation des ondes avec un terme dissipatif (1), (2) est exactement $p_\mathrm{c}(n) = 1 + 2/n$.

Notre résultat concernant l'existence globale est le suivant :

THÉORÈME 1. – *Supposons que* $p_\mathrm{c}(n) < p < n/(n-2)$, *pour* $n \geqslant 3$ *et* $p_\mathrm{c}(n) < p < \infty$, *pour* $n = 1, 2$. *Il existe* $\varepsilon_0 > 0$, *tel que le problème de Cauchy* (1), (2) *admet une solution globale* $u \in \mathrm{C}\big(\mathbb{R}_+, \mathrm{H}^1(\mathbb{R}^n)\big)$, $u_t \in \mathrm{C}\big(\mathbb{R}_+, \mathrm{L}^2(\mathbb{R}^n)\big)$, *pour chaque* $\varepsilon < \varepsilon_0$.

Notre résultat de l'explosion est un complément au théorème 1, excepté en ce qui concerne le cas critique $p_\mathrm{c}(n)$.

THÉORÈME 2. – *Supposons que* $1 < p < 1 + 2/n$. *Si l'on admet que* $c_i \equiv \int u_i(x)\,\mathrm{d}x > 0$, $i = 0, 1$, *la solution de* (1), (2) *explose dans un temps fini, quelque petit que puisse être* $\varepsilon > 0$.

Il importe de souligner que l'exposant critique $p_\mathrm{c}(n)$ pour l'équation des ondes avec un terme dissipatif (1), (2) coïncide avec l'exposant critique $p_1(n)$ de Fujita, pour l'équation de chaleur. Cela démontre que le terme dissipatif modifie radicalement le comportement asymptotique des solutions de l'équation des ondes. Autrement dit, celles-ci paraissent se comporter davantage comme des solutions de l'équation de chaleur en temps long.

Dans cette étude, nous observons par ailleurs un autre phénomène intéressant, dû à la présence du terme dissipatif : le support de la solution de (1), (2) pour $p > p_\mathrm{c}(n)$ est fortement compressé par le terme dissipatif, de sorte que la solution est concentrée dans un domaine bien plus petit que $|x| \leqslant t + K$ (la constante K étant le radius du support des données initiales $|x| \leqslant K$). Plus précisément, on démontre qu'en dehors du domaine $\mathrm{B}(t^{1/2+\delta})$, où $\delta > 0$, l'énergie totale décroît d'une manière exponentielle.

Il est important de souligner qu'aucune des techniques classiques développées pour l'exposant critique de l'équation des ondes ne fonctionne pour l'équation des ondes avec un terme dissipatif (1), (2). Aussi se voit-on obligés de rechercher d'autres techniques.

La partie la plus difficile de l'étude est celle concernant l'explosion. À notre sens, cette partie est beaucoup plus compliquée que celle concernant l'explosion de l'équation des ondes, due à la présence du terme dissipatif.

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1. Introduction

This article is concerned with the Cauchy problem for the dissipative nonlinear wave equation:

$$\Box u + u_t = |u|^p, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = \varepsilon u_0, \qquad u_t(0,x) = \varepsilon u_1, \tag{1.2}$$

where $\Box = \partial_t^2 - \Delta_x$ is the wave operator. It is assumed that $\varepsilon > 0$ and (u_0, u_1) is compactly supported data from the energy space:

$$u_0 \in \mathrm{H}^1\big(\mathbb{R}^n\big), \quad u_1 \in \mathrm{L}^2\big(\mathbb{R}^n\big), \quad \operatorname{supp} u_i \subset \mathrm{B}(K) \equiv \big\{x \in \mathbb{R}^n : |x| < K\big\}, \quad i = 0, 1.$$

We study global existence and blow up as well as the asymptotic behavior as $t \to \infty$ for solutions of (1.1), (1.2). Our interest is focussed on the so-called critical exponent $p_{\mathrm{c}}(n)$, defined as a number with the following property:

If $p_{\mathrm{c}}(n) < p$, then all small data solutions of (1.1), (1.2) are global, while if $1 < p < p_{\mathrm{c}}(n)$ all solutions of (1.1), (1.2) with data positive on average blow up, regardless of how small the data may be.

It is well known that if the damping is missing, the critical exponent for the nonlinear wave equation $\Box u = |u|^p$, is the positive root $p_0(n)$ of the equation $(n-1)p^2 - (n+1)p - 2 = 0$, where n is the space dimention. The proof of this fact, famous as Strauss' conjecture [8], took almost 20 years and the effort of many mathematicians, begining with Glassey [3], John [4], Sideris [7], Zhou [10], and ending very recently with Lindblad and Sogge [6], Georgiev, Lindblad and Sogge [2], and Tataru [9].

In this paper we solve the critical exponent problem for the wave equation (1.1), (1.2), which involves not only a source but also damping terms. We expect that the influence of the damping is powerful enough to shift the critical exponent $p_0(n)$ of the wave equation to the left, i.e., the critical exponent $p_{\mathrm{c}}(n)$ for the equation (1.1) should be strictly less than $p_0(n)$.

The main result of this work is that the critical exponent $p_{\mathrm{c}}(n)$ of the damped wave equation (1.1), (1.2) is exactly

$$p_{\mathrm{c}}(n) = 1 + 2/n.$$

Thus, we really have $p_{\mathrm{c}}(n) < p_0(n)$.

Our global existence result can be stated as follows:

THEOREM 1.1. – *Let* $p_{\mathrm{c}}(n) < p \leqslant n/(n-2)$*, for* $n \geqslant 3$*, and* $p_{\mathrm{c}}(n) < p < \infty$*, for* $n = 1, 2$*. There exists* $\varepsilon_0 > 0$ *such that the problem* (1.1), (1.2) *admits a unique global solution*

$$u \in \mathrm{C}\big(\mathbb{R}_+, \mathrm{H}^1(\mathbb{R}^n)\big), \qquad u_t \in \mathrm{C}\big(\mathbb{R}_+, \mathrm{L}^2(\mathbb{R}^n)\big),$$

for each $\varepsilon < \varepsilon_0$.

The following blowup result is complementary to Theorem 1.1 except for the critical case $p = p_{\mathrm{c}}(n)$.

THEOREM 1.2. – *Let* $1 < p < 1 + 2/n$*. If we assume that* $c_i \equiv \int u_i(x)\,\mathrm{d}x > 0,\ i = 0, 1$*, then the solution of* (1.1), (1.2) *does not exist globally for any* $\varepsilon > 0$.

It should be pointed out that the critical exponent $p_{\mathrm{c}}(n) = 1 + 2/n$ for the damped wave equation (1.1), (1.2) is exactly equal to Fujita's critical exponent $p_1(n)$ for the heat equation. This indicates that the damping term drastically changes the asymptotic behavior of solutions of the wave equation. In other words, they seem to behave more like solutions of the heat equation at large time.

In this study we have also discovered a further interesting phenomenon due to the presence of the damping term: *the support of the solution of* (1.1), (1.2) *with* $p > p_{\mathrm{c}}(n)$ *is strongly suppressed by the damping so that the solution is concentrated in a ball much smaller, then* $|x| \leqslant t + K$ (the constant K is the radius of the support of the initial data $|x| < K$). More precisely, the following estimates hold.

THEOREM 1.3. – *Let* $p > 1 + 2/n$*. Then the asymptotic behavior of any small data global solution of the Cauchy problem* (1.1), (1.2) *is given by*:

$$\big\|\mathrm{D}u(t,\cdot)\big\|_{\mathrm{L}^2(R^n \setminus \mathrm{B}(t^{1/2+\delta}))} = \mathcal{O}\big(\mathrm{e}^{-t^{2\delta}/4}\big), \quad t \to \infty, \tag{1.3}$$

that is the solution decays exponentially outside every ball $\mathrm{B}(t^{1/2+\delta})$ *with* $\delta > 0$*. Moreover, the total energy decays at the rate of the linear equation, namely*:

$$\big\|\mathrm{D}u(t,\cdot)\big\|_{\mathrm{L}^2(R^n)} = \mathcal{O}\big(t^{-n/4-1/2}\big), \quad t \to \infty. \tag{1.4}$$

It is important to note that none of the classical techniques for the critical exponent for the wave equation works for the damped wave equation (1.1), (1.2), so one is forced to find other techniques.

Section 2 is devoted to the global existence Theorem 1.1 for small data solutions. We use a weighted energy with the weight function $\mathrm{e}^{\psi(t,x)}$, where $\psi(t,x)$ behaves, roughly speaking, like $|x|^2/4(t+K)$. The explanation why this "strange" weight function occurs is very natural; see below the fundamental solution $S_2(t,x)$ of $\Box u + u_t = 0$.

Section 3 is devoted to the blow-up Theorem 1.2. This is the most difficult part which, in our opinion, is more difficult than the corresponding blow-up part for the wave equation due to the presence of the damping term. The idea of Sideris to solve the positivity problem for the wave operator $\Box$, when $n > 3$, by introducing a space-time average, is difficult to implement in our situation, on account of the very complicated explicit expression for the fundamental solution.

The main idea which we use in this region is to consider as an average the convolution $S_{2k} * Hu$, where u is the solution of the main problem (1.1), (1.2), while $S_{2k} \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^n)$ is the fundamental solution of the operator $(\Box + \partial_t)^k$. Introducing high powers $k > (n+1)/2$, we gain the both regularity and positivity of S_{2k} which are crucial to carry out the lower estimates in obtaining blow-up. The use of this convolution is paid by necessity of precised asymptotic analysis of the space-time average $S_{2k} * Hu$.

2. Global existence

The classical local existence result with regularity of the solution declared in Theorem 1.1 is well known and can be found in Strauss [8].

The following weighted energy estimates are crucial for the proof of Theorem 1.1

PROPOSITION 2.1. – *Let $u(t,x)$ be a local solution of the Cauchy problem* (1.1), (1.2) *in* $[0,T)$. *Then the following weighted energy estimate is fulfilled*:

$$(t+1)^{n/4+1/2}\big\|\mathrm{D}u(t,\cdot)\big\|_{\mathrm{L}^2} \leqslant C\varepsilon + C\Big(\max_{[0,t]}(\tau+1)^{\beta}\big\|\mathrm{e}^{\delta\psi(\tau,\cdot)}u(\tau,\cdot)\big\|_{\mathrm{L}^{2p}}\Big)^p \tag{2.1}$$

for all $t \in [0,T)$. Here the weight function $\psi(t,x) = \frac{1}{2}\big(t+K-\sqrt{(t+K)^2-|x|^2}\big)$, $|x| < t+K$, $\beta > n/4p + 1/p$ and $\delta > 0$.

PROPOSITION 2.2. – *Let $u(t,x)$ be a local solution of Cauchy problem* (1.1), (1.2) *in* $[0,T)$. *Then for all $t \in [0,T)$ the following estimate holds*:

$$\big\|\mathrm{e}^{\psi(t,\cdot)}\mathrm{D}u(t,\cdot)\big\|_{\mathrm{L}^2} \leqslant C\varepsilon + C\Big(\max_{[0,t]}(\tau+1)^{\delta}\big\|\mathrm{e}^{\gamma\psi(\tau,\cdot)}u(\tau,\cdot)\big\|_{\mathrm{L}^{p+1}}\Big)^{(p+1)/2}, \tag{2.2}$$

where $\psi(t,x) = \frac{1}{2}\big(t+K-\sqrt{(t+K)^2-|x|^2}\big)$, $|x| < t+K$, $\gamma > 2/(p+1)$ and $\delta > 0$.

Proposition 2.3 is fulfilled by any function $u(t,x)$ from the energy space with compact support.

PROPOSITION 2.3. – *Let $\theta(q) = n(1/2 - 1/q)$, $0 \leqslant \theta(q) \leqslant 1$ and let $0 < \sigma \leqslant 1$. Let $u(t,x)$ be a function from the energy space and $\operatorname{supp} u \subset \mathrm{B}(t+K)$, then we have*

$$\big\|\mathrm{e}^{\sigma\psi(t,\cdot)}u(t,\cdot)\big\|_{\mathrm{L}^q} \leqslant C(t+K)^{(1-\theta(q))/2}\big\|\nabla u(t,\cdot)\big\|_{\mathrm{L}^2}^{1-\sigma}\big\|\mathrm{e}^{\psi(t,\cdot)}\nabla u(t,\cdot)\big\|_{\mathrm{L}^2}^{\sigma}, \tag{2.3}$$

where $\psi(t,x) = \frac{1}{2}\big(t+K-\sqrt{(t+K)^2-|x|^2}\big)$, $|x| < t+K$.

3. Blow-up

The proof of the blow-up Theorem 1.2 is split into two parts. In the first part, namely $1 < p < 1 + 1/n$, we use an average $F(t) = \int u(t,x)\,\mathrm{d}x$, where u is the local solution of (1.1), (1.2) and show that

$$\ddot{F}(t) + \dot{F}(t) \geqslant \mathrm{C}(t+K)^{-n(p-1)}\big|F(t)\big|^p. \tag{3.1}$$

That F is actually forced to blow up is seen after applying the following result.

PROPOSITION 3.1. – *Let $A > -1$ and $r > 0$. Assume that $F(t)$ is a twice continuously differentiable solution of the inequality*

$$\ddot{F}(t) + \dot{F}(t) \geqslant \mathrm{C}(t+K)^{A}\big|F(t)\big|^{1+r}, \quad t > 0, \tag{3.2}$$

such that $F(0) > 0$ and $\dot{F}(0) > 0$. Then $F(t)$ blows up in finite time.

Let us turn to the second case $1 + 1/n \leqslant p < 1 + 2/n$. Here we achieve a crucial improvement of (3.1) by showing that F satisfies the strong lower estimate given below. To derive such an estimate we consider as a space-time average the convolution $S_{2k} * Hu$, where $S_{2k} \in \mathcal{D}'(\mathbb{R}^{1+n})$ is the fundamental solution of $(\Box + \partial_t)^k$ supported in the forward light cone $C_+ = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n\colon |x| < t\}$. We will use fundamental solutions S_{2k} with $k \geqslant (n+1)/2$, taking advantage of the fact that those are positive continuous functions. We state the following lower estimate for F.

PROPOSITION 3.2. – *Let the assumptions of Theorem* 1.2 *hold. Then, for each fixed B, we have*

$$F(t) \geqslant C_B(t+K)^B, \quad t \geqslant 0, \tag{3.3}$$

where $C_B > 0$.

To prove this estimate we need a precise asymptotic analysis of the convolution $S_{2k} * Hu$, collected in the following lemmas.

LEMMA 3.1. – *The fundamental solution S_{2k} is given by*

$$S_{2k}(t,x) = \begin{cases} \dfrac{\mathrm{e}^{-t/2}\big(\sqrt{t^2-|x|^2}\big)^{k-(n+1)/2} I_{k-(n+1)/2}\big(\frac{1}{2}\sqrt{t^2-|x|^2}\big)}{\pi^{(n-1)/2}2^n\Gamma(k)}, & (t,x) \in C_+, \\ 0, & (t,x) \notin C_+, \end{cases} \tag{3.4}$$

where Γ and $I_{k-(n+1)/2}$ are the gamma function and modified Bessel function respectively.

The next result is an integral identity from which we derive the lower bound on $S_{2k} * Hu$.

LEMMA 3.2. – *Let u be a local solution of* (1.1), (1.2). *For $k \geqslant (n+3)/2$, we have*

$$S_{2(k-1)} * Hu = \varepsilon\partial_t S_{2k}(t) * u_0 + \varepsilon S_{2k}(t) * (u_0 + u_1) + S_{2k} * H|u|^p, \tag{3.5}$$

where $S_{2k}(t) \equiv S_{2k}(t,\cdot)$.

The asymptotics of the linear term in (3.5) is given next.

LEMMA 3.3. – *Let $k \geqslant (n+3)/2$. Then, for $|x| < 1$, we have the following uniform asymptotics, as $t \to \infty$:*

$$S_{2k}(t) * (u_0 + u_1) = \frac{c_0 + c_1}{2^n\pi^{n/2}\,\Gamma(k)} t^{k-(n+2)/2} + \mathcal{O}\big(t^{k-(n+4)/2}\big),$$
$$\partial_t S_{2k}(t) * u_0 = \mathcal{O}\big(t^{k-(n+4)/2}\big).$$

The last estimate is an upper bound on the L^q norm of S_{2k}.

LEMMA 3.4. – *Let* $k \geqslant (n+1)/2$ *and* $1 \leqslant q < \infty$. *Then the following estimate holds*:

$$\|S_{2k}\|_{\mathrm{L}^q([0,t]\times R^n)} \leqslant Ct^{k+1/q-1-n/(2q')}, \quad q' = q/(q-1),$$

for each $t \geqslant 1$.

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