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EXISTENCE FOR A NONLINEAR WAVE EQUATION WITH DAMPING AND SOURCE TERMS

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1. Introduction

We study the Cauchy problem

$$u_{tt} - \Delta u + Q(t, x, u_t) = f(x, u), \quad u(0) = u_0,\ u_t(0) = u_1, \tag{1.1}$$

in $(0, T) \times \mathbb{R}^n$, where $0 < T \leq \infty$, $n \geq 1$. The data u_0 and u_1 are in the energy space and are compactly supported in $\mathbb{R}^n$. The terms Q and f are nonlinear damping and source terms respectively.

Problems like (1.1) have been studied in many papers. For the case $Q \equiv 0$ see [3], [7], [11], [12], [20], [30]; for the case $f \equiv 0$ see [8], [10]; when both terms Q and f interact, see [6], [9], [13], [14], [15], [18], [19], [25], [26], [27], [31], [32], [33].

For the sake of simplicity we illustrate our results by considering the model power–like equation with damping term

$$Q(x, v) = \sigma(x)|v|^{m-2}v, \tag{1.2}$$

where $m > 1$, σ is a measurable function such that $c_1 \leq \sigma(x) \leq c_2$ on $\mathbb{R}^n$, c_1, c_2 are positive constants, and the source term is in the form

$$f(x, u) = \mu_1(x)|u|^{p-2}u + \mu_2(x)|u|^{q-2}u, \tag{1.3}$$

with $p > 2$, $1 < q < p$, and $\mu_i \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$, $i = 1, 2$.

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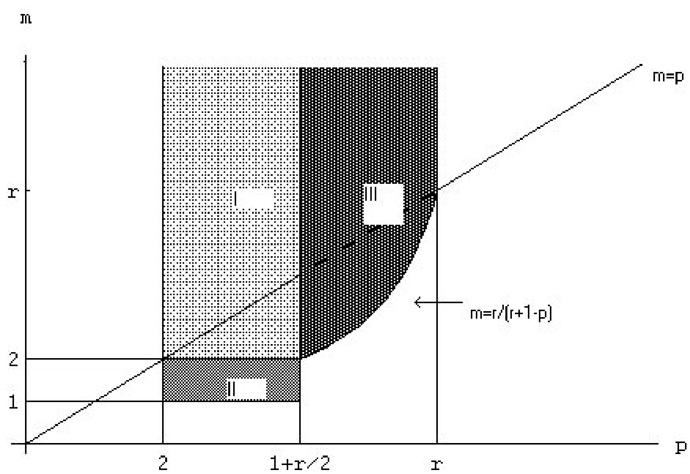

FIGURE 1. The three regions I, II and III in the (p, m) plane, for which we prove local existence and uniqueness in the case $q \geq 2$, given by

$$\begin{aligned} \mathrm{I} =& \{(p,m) \in \mathbb{R}^2 : 2 < p < 1 + r/2, \quad m \geq 2\}, \\ \mathrm{II} =& \{(p,m) \in \mathbb{R}^2 : 2 < p < 1 + r/2, \quad 1 < m < 2\}, \\ \mathrm{III} =& \{(p,m) \in \mathbb{R}^2 : 1 + r/2 \leq p < r, \quad m > r/(r+1-p)\}. \end{aligned}$$

Local existence for solutions of the equation (1.1) in a bounded domain Ω with zero Dirichlet boundary condition was proved in [6], for $\sigma \equiv \mu_1 \equiv 1$ and $\mu_2 \equiv 0$, where the exponents of nonlinearity m and p satisfy the conditions

$$m \geq 2, \qquad \text{and} \quad 2 < p \leq 1 + r/2, \tag{1.4}$$

with $r = 2n/(n-2)$ for $n \geq 3$ and $r = \infty$ for $n = 1, 2$. The above region for m and p is the right closure of the region I in the plane (p, m) in Figure 1. The particular cut (1.4) for exponents of nonlinearities is assumed in all previously quoted papers where the interaction between source and damping terms is studied. The goal of this paper is to take more careful advantage of the presence of the damping term, namely to clarify the smoothing effect of the damping (which will be manifested in particular in the region III), as well as to consider the influence of sublinear damping terms (region II). We are able to enlarge the local existence region I to the region $\mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$; more precisely, we have the following existence result for the model power–like equation:

Theorem 1. *Let Q and f satisfy* (1.2) *and* (1.3)*, and let $(p,m) \in I \cup II \cup III$, namely*

$$m > 1, \quad 2 < p < 1 + r/2 \quad \text{or} \quad m > r/(r+1-p), \quad 1 + r/2 \leq p < r. \quad (1.5)$$

Moreover, assume that $1 < q < p$, and let the data be compactly supported and in the energy space $u_0 \in H^1(\mathbb{R}^n)$, $u_1 \in L^2(\mathbb{R}^n)$. Then, there is a weak solution of (1.1) *with the properties*

$$u \in C([0,T]; H^1(\mathbb{R}^n)), \quad u_t \in C([0,T]; L^2(\mathbb{R}^n)) \quad (1.6)$$

$$u_t \in L^m((0,T) \times \mathbb{R}^n), \quad (1.7)$$

for $T > 0$ small enough.

Moreover, when $\mu_2 \equiv 0$, and exponents $(p,m) \in \overline{I_r} \cup \overline{II_r}$, where

$$\overline{I_r} = \{(p,m) \in \mathbb{R}^2 : 2 < p \leq 1 + r/2, \quad m \geq 2\},$$
$$\overline{II_r} = \{(p,m) \in \mathbb{R}^2 : 2 < p \leq 1 + r/2, \quad 1 < m < 2\}$$

respectively are the right closure of I and II, a unique solution of (1.1) *exists, with the same regularity.*

Remark 1. Theorem 1 is a particular case of our general local existence result (see Theorem 7 below) for the equation (1.1), with damping $Q = Q(t,x,u_t)$ and source $f = f(x,u)$, under mild assumptions which will be presented in the sequel.

As an extension of the above result we have the following global existence result, for the case $p \leq m$.

Theorem 2. *Let Q and f satisfy* (1.2) *and* (1.3) *and $2 < p < r$, $1 < q < p \leq m$. Then for any compactly supported data in the energy space $u_0 \in H^1(\mathbb{R}^n)$, $u_1 \in L^2(\mathbb{R}^n)$ the Cauchy problem* (1.1) *has a global solution with the regularity* (1.6)–(1.7)*.*

Remark 2. Once again our global existence theorem is for more general nonlinearities Q and f compared with the ones considered in Theorem 2. The general global existence Theorem 8 will be formulated and proved in the sequel. Let us mention that this global existence theorem is a generalization of the global existence theorem in [27].

Actually, with essentially not power like damping and source terms, namely, with $Q(u_t)$ and $f(x,u)$ being derivatives of $N-$functions, then under appropriate conditions an existence – regularity theorem could be proved.

Remark 3. Further applications of our general existence–regularity theorem, concerning global existence and energy decay, will be presented in a forthcoming paper [29]. We shall present several blow-up type applications of the general local existence Theorem 7 in another paper [28].

Following the idea of the proof of the general global existence Theorem 8, we can derive a global existence result for any compactly supported data in the energy space, for the Cauchy problem with decaying potential

$$\begin{cases} u_{tt} - \Delta u + \alpha(t)|u_t|^{m-2}u_t = f(x,u), & \text{in } (0,\infty)\times\mathbb{R}^n, \\ u(0) = u_0, \qquad u_t(0) = u_1, \end{cases} \tag{1.8}$$

where $\alpha(t) = 1/(1+t)^\beta$, $\beta > 0$, and f satisfies (1.3). We state this as

Corollary 1. *Let $m \geq 2$. Then for any compactly supported data in the energy space the problem* (1.8) *has a global weak solution with the regularity* (1.6)–(1.7).

Remark 4. This result does not depends on the range of the potential. Even a very short range potential does not effect the global existence result.

Now we emphasize some of the essential difficulties to be faced and the way to overcome them.

The main difficulty in the region II ($1 < m < 2$) is the lack of an overall Lipschitz behavior for the damping, which badly effects the proof of the finite propagation speed in this region. Indeed a Lipschitz condition is a basic tool for Strauss' argument [25] proving a finite speed of propagation. For the sake of simplicity, we consider the problem when the damping is in the simple form $Q(t,x,v) = |v|^{m-2}v$. We approximate Q uniformly with conveniently chosen Lipschitz functions such that the approximate equation

$$u_{tt} - \Delta u + Q_\varepsilon(u_t) = g(t,x) \tag{1.9}$$

has a finite speed of propagation. Here $g(t,x)$ is a conveniently chosen replacement for the nonlinear source $f(x,u)$, with support in the light cone. To carry out this procedure we need an existence–regularity result for (1.9). Here the main difficulty appears, namely we are no longer able to stay in the Lebesgue spaces used by Strauss. To show that this is the case, we present a list of requirements for Q_ε which would follow from Lipschitzianity and a finite speed of propagation for (1.9). From the finite speed of propagation it follows that $u = 0$ satisfies (1.9) outside the light cone, i.e., the first requirement is $Q_\varepsilon(0) = 0$. From this and a Lipschitz condition we get the next requirement, namely, the estimate

$$|Q_\varepsilon(v)| \leq c_1^\varepsilon |v| \tag{1.10}$$

for some constant c_1^ε. Then to derive an a priori estimate for u_t^ε, where u^ε is a solution of the approximate problem (1.9) in some Lebesgue space, we have to ask that

$$Q_\varepsilon(v)v \geq c_2^\varepsilon |v|^\mu \tag{1.11}$$

for some $\mu > 1$. Due to the uniformity of the approximation, μ cannot be different from m, which is clearly impossible because (1.10) and (1.11) lead to a contradiction when $\mu = m < 2$. Thus, we realize that it is not possible to stay within any Lebesgue space. Nevertheless we are still able to derive an estimate

$$Q_\varepsilon(v)v \geq c_2^\varepsilon \Phi_\varepsilon(v),$$

where Φ_ε is a conveniently chosen N-function. In this way, we naturally come to the idea of embedding u_t^ε in the corresponding Orlicz space $L^{\Phi_\varepsilon}((0,T) \times\mathbb{R}^n)$. Since Q_ε is no longer power–like, we need to prove a new existence–regularity result for (1.9), where the damping Q_ε is essentially the derivative of an N – function (under mild conditions). This result, together with some continuity results for Orlicz spaces, and a density lemma in the intersection of Orlicz and Lebesgue spaces (which we prove in Appendix B) closes the circle.

The smoothness effect of the damping is mainly manifested in the region III. Indeed, due to the presence of the damping term, we are able to obtain a space – time estimate for u_t which plays a basic role in controlling the kinetic energy of the solution. This, together with an appropriate compactness result, allows us to use the Schauder fixed point theorem. On the other hand, for the region $\overline{\mathrm{I}_r} \cup \overline{\mathrm{II}_r}$, we can improve this result by applying the contraction principle and thus obtain uniqueness as well.

2. Global existence with a given forcing term

This section is devoted to global existence results for the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + Q(t,x,u_t) = g(t,x), & \text{in } (0,T)\times\mathbb{R}^n, \\ u(0) = u_0, \qquad u_t(0) = u_1. \end{cases} \tag{2.1}$$

These results will be essentially used in the sequel.

We assume that Q is a Caratheodory function on $(0,T)\times\mathbb{R}^n$, i.e., that $Q = Q(t,x,v)$ is continuous in v for almost all $(t,x) \in (0,T)\times\mathbb{R}^n$ and measurable in (t,x) for all $v \in \mathbb{R}$. Moreover, let the following assumptions hold:

(Q1) Q is increasing in v for almost all $(t,x) \in (0,T)\times\mathbb{R}^n$;

(Q2) there exist $m > 1$ and $c_1 > 0$ such that

$$|Q(t,x,v)| \le c_1 |v|^{m-1}$$

for almost all $(t,x) \in (0,T) \times \mathbb{R}^n$ and all $v \in \mathbb{R}$;

(Q3) there is $c_2 > 0$ such that

$$Q(t,x,v)v \ge c_2 |v|^m$$

for almost all $(t,x) \in (0,T) \times \mathbb{R}^n$ and all $v \in \mathbb{R}$.

The first result of this section generalizes the Lions and Strauss bounded domain global existence result [16, Theorem 6.1] to the Cauchy problem in $\mathbb{R}^n$ and to nonlinearities satisfying (Q1)–(Q3).

Theorem 3. *Let Q satisfy* (Q1)*–*(Q3)*, let the data satisfy $u_0 \in H^1(\mathbb{R}^n)$, $u_1 \in L^2(\mathbb{R}^n)$, and suppose $g(t,x) \in L^2((0,T) \times \mathbb{R}^n)$. Then there is a unique global weak solution of* (2.1) *such that*

$$u \in C([0,T]; H^1(\mathbb{R}^n)), \quad u_t \in C([0,T]; L^2(\mathbb{R}^n)) \cap L^m((0,T) \times \mathbb{R}^n).$$

Remark 5. The solution also satisfies the energy identity

$$\tfrac{1}{2}\|u_t(s)\|_2^2 + \tfrac{1}{2}\|\nabla u(s)\|_2^2 \Big|_0^t + \int_{(0,t)\times\mathbb{R}^n} Q(t,x,u_t)u_t = \int_{(0,t)\times\mathbb{R}^n} g u_t,$$

as follows directly from Strauss' argument [24, Theorem 4.2].

Proof. We start by reducing the Cauchy problem to an initial–boundary value problem (IBVP) in $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$. First we approximate the data and the forcing term $g(t,x)$ with conveniently chosen functions in B_ρ. Let $\eta \in C^\infty(\mathbb{R})$ be a cut–off function such that $\eta(s) = 0$ for $|s| \ge 1$ and $\eta(s) = 1$ for $|s| \le 1/2$. Put $\eta_\rho(x) = \eta(4|x|^2/\rho^2)$. Then we define

$$u_{0\rho} = \eta_\rho u_0 \in H_0^1(B_\rho), \quad u_{1\rho} = \eta_\rho u_1 \in L^2(B_\rho), \quad g_\rho = \eta_\rho g \in L^2((0,T) \times B_\rho). \tag{2.2}$$

Clearly

$$\nabla u_{0\rho}(x) = \nabla u_0(x)\eta_\rho(x) + u_0(x)\eta'(4|x|^2/\rho^2)8x/\rho^2.$$

Next, by the Lebesgue dominated convergence theorem, we get

$$u_{0\rho} \to u_0 \in H^1(\mathbb{R}^n), \;\; u_{1\rho} \to u_1 \in L^2(\mathbb{R}^n), \;\; g_\rho \to g \in L^2((0,T) \times \mathbb{R}^n). \tag{2.3}$$

Now consider the perturbed initial–boundary value problem

$$\begin{cases} u_{tt} - \Delta u + Q(t,x,u_t) = g_\rho, & \text{in } (0,T) \times B_\rho, \\ u = 0 & \text{in } (0,T) \times \partial B_\rho, \\ u(0) = u_{0\rho}, \qquad u_t(0) = u_{1\rho}. \end{cases} \tag{2.4}$$

Since the nonlinear damping term $Q(t,x,u_t)$ could possibly contain a non–Lipschitz part, we are not able to use the standard Faedo–Galerkin procedure which requires a Lipschitz argument to obtain the solution of the corresponding ordinary differential equation. Instead, we are forced to use Carathéodory's Theorem (see [5]) to insure the existence of a solution of the corresponding ordinary differential systems. We complete the proof of the global existence result for (2.4) in the standard way, obtaining the same regularity for the solution u^ρ, i.e.,

$$\begin{aligned} &u^\rho \in L^\infty(0,T;H_0^1(B_\rho)), \quad u_t^\rho \in L^\infty(0,T,L^2(B_\rho)) \cap L^m((0,T)\times B_\rho), \\ &Q(t,x,u_t^\rho) \in L^{m'}((0,T)\times B_\rho). \end{aligned} \tag{2.5}$$

Moreover, the solution u^ρ of (2.4) satisfies the energy identity. As a consequence of this and the convergence (2.3) of $u_{0\rho}$, $u_{1\rho}$ and g_ρ, we obtain

$$\sup_{t\in[0,T)} \|u_t^\rho(t)\|_2^2 + \|\nabla u^\rho(t)\|_2^2 \le k_1, \tag{2.6}$$

$$\int_{(0,T)\times\mathbb{R}^n} Q(\cdot,\cdot,u_t^\rho)u_t^\rho \le k_1, \tag{2.7}$$

where the constant k_1 is independent of ρ. Moreover, since

$$\|u^\rho(t)\|_2 \le \|u_{0\rho}\|_2 + \int_0^T \|u_t^\rho\|_2$$

for all $t\in[0,T]$, from the convergence $(2.3)_1$ and from the estimate (2.6) we get

$$\sup_{t\in[0,T)} \|u^\rho(t)\|_2 \le k_2, \tag{2.8}$$

where the constant k_2 is independent of ρ. Then, using assumptions (Q1) and (Q2) together with the estimates (2.6)–(2.8), we obtain (up to a subsequence) the convergence

$$\begin{aligned} &u^\rho \to u \quad \text{weakly}^* \text{ in } L^\infty(0,T;H^1(\mathbb{R}^n)), \\ &u_t^\rho \to u_t \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\mathbb{R}^n)) \text{ and weakly in } L^m((0,T)\times\mathbb{R}^n), \\ &Q(\cdot,\cdot,u_t^\rho) \to \chi \quad \text{weakly in } L^{m'}((0,T)\times\mathbb{R}^n), \end{aligned} \tag{2.9}$$

as $\rho\to\infty$. We shall prove that u is a solution in a distribution sense of the equation

$$u_{tt} - \Delta u + \chi - g = 0 \qquad \text{in } H^{-1}(\mathbb{R}^n) + L^{m'}(\mathbb{R}^n), \tag{2.10}$$

where the space $H^{-1}(\mathbb{R}^n)$ is the dual of $H^1(\mathbb{R}^n)$. First, we multiply equation (2.4) by $\phi \in C_c^\infty(0,T)$ and integrate over $[0,T)$, to get

$$\int_0^T -u_t^\rho \phi' + [-\Delta u^\rho + Q(\cdot,\cdot,u_t^\rho) - g_\rho]\phi \, ds = 0, \tag{2.11}$$

where an integration by parts was used. Then we fix $w \in C_c^\infty(\mathbb{R}^n)$ and chose ρ sufficiently large so that supp w is compactly included in B_ρ. Multiplying (2.11) by w and integrating over $\mathbb{R}^n$, we derive

$$\int_{(0,T)\times\mathbb{R}^n} -u_t^\rho w\phi' + [\nabla u^\rho \nabla w + Q(t,x,u_t^\rho)w - g_\rho w]\phi = 0. \tag{2.12}$$

Finally, due to the density of $C_c^\infty(\mathbb{R}^n)$ in $H^1(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$, and using the regularity (2.5), we see that (2.12) is fulfilled for all test functions $w \in H^1(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$. Passing to the limit as $\rho \to \infty$ in (2.12), and using the convergences (2.9) and $(2.3)_3$, we get

$$\int_0^T -u_t\phi' + [-\Delta u + \chi - g]\phi = 0 \qquad \text{in } H^{-1}(\mathbb{R}^n) + L^{m'}(\mathbb{R}^n).$$

It remain to prove that $\chi = Q(\cdot,\cdot,u_t)$ and $u(0) = u_0$, $u_t(0) = u_1$. Using the assumption (Q1) and the celebrated Lions–Strauss monotonicity argument we obtain that $\chi = Q(\cdot,\cdot,u_t)$. The relations $u(0) = u_0$, $u_t(0) = u_1$, the energy identity and the uniqueness of u are derived exactly as in [16] and [24]. □

We need one more global existence result for equation (2.1) with more regular data. For this we shall require some assumptions on the derivative of the damping term in the case $m \geq 2$. More precisely, we assume that Q_t and that Q_v are Caratheodory functions, and

(Q4) there exists $c_3 \geq 0$ such that

$$|Q_t(t,x,v)| \leq c_3|v|^{m-1}$$

for almost all $(t,x) \in (0,T) \times \mathbb{R}^n$ and all $v \in \mathbb{R}$;

(Q5) there exists $\chi \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\psi \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ such that

$$|Q_v(t,x,v)| \leq \chi(x)\psi(v)$$

for almost all $(t,x) \in (0,T) \times \mathbb{R}^n$ and all $v \in \mathbb{R}$;

(Q6) there is $c_4 > 0$ such that

$$Q_v(t,x,v) \geq c_4|v|^{m-2}$$

for almost all $(t,x) \in (0,T) \times \mathbb{R}^n$ and all $v \in \mathbb{R}$.

We are now able to state

Theorem 4. *Assume that the assumptions* (Q1)*–*(Q6) *hold, that* $m \geq 2$, *and*

$$u_0 \in H^2(\mathbb{R}^n), \;\; u_1 \in H^1(\mathbb{R}^n) \cap L^{2(m-1)}(\mathbb{R}^n), \;\; g, g_t \in L^2((0,T) \times \mathbb{R}^n). \tag{2.13}$$

Let u *be the solution of* (2.1) *given by Theorem* 3. *Then in addition to the regularity assured by Theorem* 3, *we have*

$$u_t \in L^\infty(0,T; H^1(\mathbb{R}^n)), \qquad \textit{and} \quad u_{tt} \in L^\infty(0,T; L^2(\mathbb{R}^n)). \tag{2.14}$$

Remark 6. The corresponding regularity cannot be obtained for region II, namely when the growth rate m of the damping is between 1 and 2. The main reason for this is the lack of a Lipschitz condition for the damping term Q in this region.

Proof. To derive the higher regularity of the solution u, we start with the calculation of derivatives for $u_{0\rho}$ and g_ρ (see (2.2)). Namely, we have [1]

$$\begin{aligned} D^2 u_{0\rho}(x) &= \eta_\rho(x) D^2 u_0(x) + 16/\rho^2 \eta'\left(4|x|^2/\rho^2\right) \nabla u_0(x) \otimes x \\ &\quad + 8/\rho^2 u_0(x) \left[\eta'\left(4|x|^2/\rho^2\right) I_n + 8/\rho^2 \eta''\left(4|x|^2/\rho^2\right) x \otimes x\right] \end{aligned}$$

and $(g_\rho)_t = \eta_\rho g_t$. With the help of (2.13) we can improve the convergence (2.3) in the following way:

$$u_{0\rho} \to u_0 \;\text{ in } H^2(\mathbb{R}^n), \quad u_{1\rho} \to u_1 \quad \text{in } H^1(\mathbb{R}^n) \cap L^{2(m-1)}(\mathbb{R}^n), \tag{2.15}$$

$$g_\rho \to g \;\text{ in } L^2((0,T) \times \mathbb{R}^n), \quad (g_\rho)_t \to g_t \quad \text{in } L^2((0,T) \times \mathbb{R}^n). \tag{2.16}$$

Next we need some estimates for the solution u_ν in a modified Faedo–Galerkin approximation procedure, namely for the equations

$$\begin{cases} (u_\nu''(t), w_j) + (\nabla u_\nu(t), \nabla w_j) + (Q(t, \cdot, u_\nu'(t)), w_j) = (g_\rho(t), w_j), \\ u_\nu(0) = u_{0\nu}, \qquad u_\nu'(0) = u_{1\nu}, \end{cases} \tag{2.17}$$

where $j = 1, \ldots, \nu$, and $w_j \in C^\infty(\overline{B_\rho})$ are the eigenfunctions of the Laplacian on B_ρ. In the above, we choose $u_{0\nu}, u_{1\nu} \in W_\nu := \operatorname{span}[w_1, \ldots, w_\nu]$ and approximate the initial data in the sense that

$$u_{0\nu} \to u_{0\rho} \quad \text{in } H^2(B_\rho), \quad u_{1\nu} \to u_{1\rho} \quad \text{in } H_0^1(B_\rho) \cap L^{2(m-1)}(B_\rho). \tag{2.18}$$

Following the ideas of [17], we get the estimates

$$\|u_\nu''(0)\|_2 \leq \|\Delta u_{0\nu}\|_2 + \|g_\rho(0)\|_2 + c_1 \|u_{1\nu}\|_{2(m-1)}^{m-1}, \tag{2.19}$$

$$\sup_{t \in (0,T)} \|u_\nu''(t)\|_2^2 + \|\nabla u_\nu'(t)\|_2^2$$

[1] $x \otimes y$ denotes the standard tensor product in $\mathbb{R}^n$, and I_n the identity matrix in $\mathbb{R}^n$.

$$\leq \text{Const.}\left(|||u'_\nu|||_m^m + |||(g_\rho)_t|||_2^2 + \|u''_\nu(0)\|_2^2 + \|\nabla u_{1\nu}\|_2^2 \right). \tag{2.20}$$

Here and in the sequel we use the notation $|||h|||_s$ for the time – space norm $(\int_{(0,T)\times\mathbb{R}^n} |h|^s)^{1/s}$. Moreover we have the convergence conditions

$$u''_\nu \to (u^\rho)_{tt} \qquad \text{weakly}^* \text{ in } L^\infty(0,T,L^2(B_\rho)) \tag{2.21}$$

$$u'_\nu \to (u^\rho)_t \qquad \text{weakly}^* \text{ in } L^\infty(0,T,H_0^1(B_\rho)). \tag{2.22}$$

Passing to the limit as $\nu \to \infty$ in (2.19) and using (2.18), we have

$$\limsup_\nu \|u''_\nu(0)\|_2^2 \leq \|\Delta u_{0\rho}\|_2 + \|g_\rho(0)\|_2 + c_1\|u_{1\rho}\|_{2(m-1)}^{m-1}. \tag{2.23}$$

By the energy identity for u_ν, the Gronwall lemma, and passage to the limit as $\nu \to \infty$, we obtain

$$\limsup_\nu \int_{(0,T)\times B_\rho} Q(\cdot,\cdot,u'_\nu)u'_\nu \leq \text{Const.}\left(\|u_{1\rho}\|_2^2 + \|\nabla u_{0\rho}\|_2^2 + |||g_\rho|||_2^2\right). \tag{2.24}$$

Then, using assumption (Q3) we can rewrite the previous estimate in the form

$$\limsup_\nu |||u'_\nu|||_m^m \leq \text{Const.}\left(\|u_{1\rho}\|_2^2 + \|\nabla u_{0\rho}\|_2^2 + |||g_\rho|||_2^2\right). \tag{2.25}$$

Passing to the limit in (2.20), using the convergences (2.16), (2.18), (2.21), (2.22) and the estimates (2.23), (2.25), we obtain

$$\begin{aligned}\sup_{t\in(0,T)} \|u^\rho_{tt}(t)\|_2^2 + \|\nabla u^\rho_t(t)\|_2^2 \leq \text{Const.}\Big(&\|u_{0\rho}\|_{H^2(\mathbb{R}^n)} + \|u_{1\rho}\|_{H^1(\mathbb{R}^n)} \\ &+ \|u_{1\rho}\|_{2(m-1)}^{m-1}\|g_\rho(0)\|_2^2 + |||g_\rho|||_2^2 + |||(g_\rho)_t|||_2^2\Big),\end{aligned} \tag{2.26}$$

where we have also used the lower semi–continuity of the norm in the space $L^\infty(0,T;L^2(\mathbb{R}^n))$ with respect to the weak* topology. By the convergence (2.16) together with the inequality (2.26) we get the estimates

$$\sup_{t\in(0,T)} \|u^\rho_{tt}(t)\|_2 \leq \text{Const.}, \qquad \text{and} \qquad \sup_{t\in(0,T)} \|\nabla u^\rho_t(t)\|_2 \leq \text{Const.}, \tag{2.27}$$

where the constants are ρ – independent. Then, passing to the limit as $\rho \to \infty$, we can extend the convergence (2.9) with the further results

$$u^\rho_{tt} \to u_{tt}, \quad \text{and} \quad \nabla u^\rho_t \to \nabla u_t \qquad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\mathbb{R}^n)). \tag{2.28}$$

This completes the proof. □

Remark 7. We can get more regularity for the solution u than is assured by Theorem 4 by assuming additional properties for the damping term and for $g(t,x)$. Namely, provided that

$$|\nabla_x Q(t,x,v)| \le c_5 |v|^{m-1}$$

is fulfilled for almost all $(t,x) \in (0,T) \times \mathbb{R}^n$ and all $v \in \mathbb{R}$, and $g \in L^2(0,T;H^1(\mathbb{R}^n))$, the solution u of equation (2.1) then has the additional regularity $u \in L^\infty(0,T;H^2(\mathbb{R}^n))$. We omit the proof of this remark since it can be performed using arguments similar to those of Theorem 3 and 4.

The above existence–regularity results are basically related with power–type nonlinearities. In Section 3 we shall prove an existence–regularity result for essentially different types of nonlinearities.

3. Finite speed of propagation

In this section we prove the finite propagation speed for solutions of equation (2.1) when the growth rate m of the damping term satisfies $1 < m < 2$.

As mentioned in the introduction, since the damping term obeys a Lipschitz condition when $m \ge 2$, the existence of a finite propagation speed for this case is almost a direct consequence of the argument of Strauss (see [25]), adapted for weak solutions.

In the region $1 < m < 2$ the main difficulty is the lack of Lipschitz behavior for the damping. We overcome this difficulty by approximating the damping term with Lipschitz continuous functions Q_ε. This approximation however causes other difficulties since we are not able to prove that the time derivative of the solution of the approximate problems is in the Lebesgue space L^m. On the other hand we are able to show that the time derivative is in a conveniently chosen Orlicz space. See Proposition 1 and Theorem 5 below. For this approximating procedure we need to construct sufficiently smooth functions which are quadratic near 0 and behave like t^m for sufficiently large t. The construction is given in the following lemma.

Lemma 1. *Let $1 < m < 2$. For each $\varepsilon > 0$ there is a strictly convex function $\Phi_\varepsilon \in C^2([0,\infty))$ such that*

$$\Phi_\varepsilon(0) = \Phi'_\varepsilon(0) = 0, \tag{3.1}$$

$$\Phi_\varepsilon(t) = t^m \quad \textit{for all } t \ge \varepsilon, \tag{3.2}$$

$$\Phi_\varepsilon(t) = O(t^2) \quad \textit{as } t \to 0^+. \tag{3.3}$$

Moreover, Φ_ε satisfies the estimates

$$\Phi_\varepsilon(2t) \le c_{1\varepsilon} \Phi_\varepsilon(t) \quad \textit{for all } t \ge 0, \tag{3.4}$$

$$2l_\varepsilon \Phi_\varepsilon(t) \le \Phi_\varepsilon(l_\varepsilon t) \quad \textit{for all } t \ge 0, \tag{3.5}$$

$$\Phi'_\varepsilon(t) \le c_5 t^{m-1} \quad \textit{for all } t \ge 0 \tag{3.6}$$

$$\Phi''_\varepsilon(t)t \le c_{2\varepsilon}\Phi'_\varepsilon(t) \quad \textit{for all } t \ge 0, \tag{3.7}$$

where the constants $c_{1\varepsilon}, c_{2\varepsilon} > 0$*,* $l_\varepsilon > 1$ *may depend on* ε *but* $c_5 > 0$ *is independent of* ε*.*

Proof. We define the function Φ_ε by

$$\Phi_\varepsilon(t) = \begin{cases} t^2(a_\varepsilon t^2 + b_\varepsilon t + c_\varepsilon), & t < \varepsilon \\ t^m, & t \ge \varepsilon \end{cases} \tag{3.8}$$

where a_ε, b_ε and c_ε are conveniently chosen constants in $\mathbb{R}$ such that $\Phi_\varepsilon \in C^2([0,\infty))$, namely $a_\varepsilon = \frac{1}{2}(m-2)(m-3)\varepsilon^{m-4}$, $b_\varepsilon = -(m-2)(m-4)\varepsilon^{m-3}$, and $c_\varepsilon = \frac{1}{2}(m-3)(m-4)\varepsilon^{m-2}$. Elementary calculations using the facts that $a_\varepsilon > 0$ and $1 < m < 2$ show that Φ_ε is a strictly convex function, while (3.1)–(3.3) are trivial consequences of the definition (3.8). Properties (3.4) and (3.7) easily follow from (3.8), taking into account the behavior of the functions $\Phi''_\varepsilon(t)t/\Phi'_\varepsilon(t)$ and $\Phi_\varepsilon(2t)/\Phi_\varepsilon(t)$ near 0 and ∞. Inequality (3.6) also follows from (3.8), the homogeneity of a_ε, b_ε and c_ε and the fact that $1 < m < 2$.

The proof of (3.5) has to be divided in three different cases:

(a) large t, i.e., $t \ge \varepsilon$;
(b) intermediate t, i.e., $\varepsilon/l_\varepsilon < t < \varepsilon$;
(c) small t, i.e., $t \le \varepsilon/l_\varepsilon$.

The case (a) is trivial. For the case (b) it is sufficient to verify

$$2(a_\varepsilon t^2 + b_\varepsilon t + c_\varepsilon) \le l_\varepsilon^{m-1} t^{m-2} \tag{3.9}$$

for $t \in (\varepsilon/l_\varepsilon, \varepsilon)$, which follows from the convexity of the function $a_\varepsilon t^2 + b_\varepsilon t + c_\varepsilon$, namely

$$a_\varepsilon t^2 + b_\varepsilon t + c \le \max\{a_\varepsilon \varepsilon^2 + b_\varepsilon \varepsilon + c_\varepsilon, a_\varepsilon(\varepsilon/l_\varepsilon)^2 + b_\varepsilon(\varepsilon/l_\varepsilon) + c_\varepsilon\},$$

and the inequality $l_\varepsilon^{m-1} t^{m-2} \ge l_\varepsilon^{m-1}\varepsilon^{m-2}$ for $t \in (\varepsilon/l_\varepsilon, \varepsilon)$. Then (3.9) is fulfilled for sufficiently large l_ε. In the third case we have to verify that

$$a_\varepsilon(l_\varepsilon^3 - 2)t^2 + b_\varepsilon(l_\varepsilon^2 - 2)t + c_\varepsilon(l_\varepsilon - 2) \ge 0$$

for $t \le \varepsilon/l_\varepsilon$, which is true for l_ε sufficiently large. □

For the sake of simplicity we shall prove the main theorem of this section only for damping terms which satisfy the assumption

(Q7) for almost all $x \in \mathbb{R}^n$ and all $v \in \mathbb{R}$,

$$Q(t,x,v) = Q(x,v) = \sigma(x)|v|^{m-2}v$$

where $\sigma \in L^\infty(\mathbb{R}^n)$ is such that $\inf_{\mathbb{R}^n} \sigma > 0$.

It is easy to see that (Q7) implies the previous assumptions (Q1)–(Q3).

Now, using the function Φ_ε constructed in Lemma 1, the approximation of the damping term can be done in the following way:

$$Q_\varepsilon(x,v) = \frac{1}{m}\sigma(x)\Phi'_\varepsilon(|v|)\text{sign } v. \tag{3.10}$$

Then $Q_\varepsilon(x,\cdot) \in C(\mathbb{R})$ for almost all $x \in \mathbb{R}^n$ and, more important, Q_ε is uniformly Lipschitz continuous in v, i.e., $(Q_\varepsilon)_v \in L^\infty(\mathbb{R}^{n+1})$. Let us note that

$$Q_\varepsilon(\cdot,v) \equiv Q(\cdot,v), \qquad \text{for } |v| \geq \varepsilon, \tag{3.11}$$

and

$$|Q_\varepsilon(x,v)| \leq |\sigma(x)|\varepsilon^{m-1}, \qquad \text{for } |v| \leq \varepsilon, \tag{3.12}$$

where to get the last inequality we used the convexity of Φ_ε. We approximate problem (2.1) by

$$\begin{cases} u_{tt} - \Delta u + Q_\varepsilon(x,u_t) = g(t,x), & \text{in } (0,T)\times\mathbb{R}^n, \\ u(0) = u_0, \qquad u_t(0) = u_1. \end{cases} \tag{3.13}$$

Unfortunately, we are not able to assure existence of a regular solution of this approximate problem because all previous existence theorems apply for nonlinearities Q which essentially dominate a power of v. Evidently, Q_ε does not satisfy this requirement, i.e., the assumption (Q3). Thus, we need a specific existence–regularity result for (3.13) (actually this result applies for more general equations, see Theorem 5 below).

The peculiarity of (3.13) is that the time derivative u_t^ε of its solution lives only in an Orlicz space, and accordingly we are led to prove regularity in Orlicz spaces.

We briefly recall some basic definitions from Orlicz space theory (see [1] or [21]). First we have to define the so–called N–functions. We say that a real valued function Φ on $[0,\infty)$ is an N–function if $\Phi(s) = \int_0^s \varphi(\eta)\,d\eta$, where $\varphi : [0,\infty) \to \mathbb{R}$ is such that

(a) $\varphi(0) = 0$, $\varphi(s) > 0$ for $s > 0$, $\lim_{s\to\infty}\varphi(s) = +\infty$;
(b) φ is nondecreasing;
(c) φ is right continuous.

The simplest example of a N–function is a power. The non–trivial example which we shall use in the sequel are functions Φ_ε constructed in Lemma 1.

The N–function $\Psi(s) = \max_{\eta \geq 0}\{s\eta - \Phi(\eta)\}$ is called the complementary function of Φ and (Φ, Ψ) a complimentary pair.

Let A be $\mathbb{R}^n$ or $(0,T) \times \mathbb{R}^N$, and Φ be a N–function. We denote by $L^\Phi(A)$ the Orlicz space on A associated to Φ in the following way

$$L^\Phi(A) = \{u \text{ is a measurable real function on } A \text{ such that } \textstyle\int_A \Phi(|u|)dx < \infty\}, \tag{3.14}$$

with norm defined by

$$\|u\|_\Phi = \inf\left\{k > 0 : \textstyle\int_A \Phi(|u|/k) \leq 1\right\}. \tag{3.15}$$

We recall the so–called Hölder inequality for Orlicz space, i.e., that given a complimentary pair (Φ, Ψ) and functions $u \in L^\Phi(A)$ and $v \in L^\Psi(A)$, we can estimate the L^1 norm of uv by

$$\left| \int_A uv \right| \leq 2\|u\|_\Phi \|v\|_\Psi. \tag{3.16}$$

Now we are able to formulate our existence and regularity result.

Proposition 1. *Assume that the data satisfy the condition $u_0 \in H^2(\mathbb{R}^n)$, $u_1 \in H^1(\mathbb{R}^n)$, $|u_1|^{m-2}u_1 \in L^2(\mathbb{R}^n)$, and $g, g_t \in L^2((0,T) \times \mathbb{R}^n)$. Then* (3.13) *has a unique weak solution u^ε in the sense that*

$$(u_t^\varepsilon(s), \phi(s))\Big|_0^t = \int_{(0,t)\times\mathbb{R}^n} u_t^\varepsilon \phi_t - \nabla u^\varepsilon \nabla \phi - Q_\varepsilon(\cdot, u_t^\varepsilon)\phi + g\phi \tag{3.17}$$

for all $t \in [0,T]$ and all test functions $\phi \in C_c^\infty(\mathbb{R}^{n+1})$. Moreover, we have

$$u^\varepsilon \in C([0,T]; H^1(\mathbb{R}^n)), \quad u_t^\varepsilon \in L^{\Phi_\varepsilon}((0,T) \times \mathbb{R}^n) \cap L^\infty(0,T; H^1(\mathbb{R}^n)), \tag{3.18}$$

$$u_{tt}^\varepsilon \in L^\infty(0,T; L^2(\mathbb{R}^n)). \tag{3.19}$$

Remark 8. In addition there holds

$$Q_\varepsilon(\cdot, u_t^\varepsilon) \in L^{\Psi_\varepsilon}((0,T) \times \mathbb{R}^n), \tag{3.20}$$

where Ψ_ε is the complementary N–function of Φ_ε.

The proof of Proposition 1 is technical and is postponed to Appendix A, together with the proof of Remark 8. The above existence and regularity result can be extended to more general damping terms, which are essentially not power–like. More precisely, the following result holds.

Theorem 5. *Assume that $\tilde{Q} \in C^1(\mathbb{R})$ is a derivative of an $N-$function, i.e., suppose that the $N-$function $\Phi(v)$ and its complimentary function $\Psi(v)$ are Δ_2-regular and moreover the following estimates hold:*

$$\tilde{Q}(v) \le K_1 v^{m-1}, \qquad \tilde{Q}'(v)v \le K_2 \tilde{Q}(v)$$

for all $v \ge 0$, where K_1, K_2 are positive constants and $m \in (1, \infty)$. Then the statements of Proposition 1 *and Remark* 8 *hold, with Q_ε, Φ_ε, Ψ_ε replaced respectively by $\tilde{Q}$, Φ, Ψ and the problem* (3.13) *by*

$$\begin{cases} u_{tt} - \Delta u + \tilde{Q}(u_t) = g(t,x), & \text{in } (0,T) \times \mathbb{R}^n, \\ u(0) = u_0, \qquad u_t(0) = u_1. \end{cases} \tag{3.21}$$

We omit the proof of Theorem 5 since it is similar to that of Proposition 1 (see Appendix A).

Now we can state the main theorem of this section.

Theorem 6. *Suppose that $1 < m < 2$, that* (Q7) *holds and the data have the following regularity*

$$u_0 \in H^2(\mathbb{R}^n), \quad u_1 \in H^1(\mathbb{R}^n), \quad |u_1|^{m-2}u_1 \in L^2(\mathbb{R}^n), \quad g \in L^2((0,T) \times \mathbb{R}^n).$$

Moreover, let the data satisfy

$$\operatorname{supp} u_0, \operatorname{supp} u_1 \subset B_R, \quad \text{and} \quad \operatorname{supp} g(t) \subset B_{R+t} \text{ for almost all } t \in [0,T], \tag{3.22}$$

for some $R > 0$. Then the solution u of equation (2.1) *has finite speed of propagation, i.e.,*

$$\operatorname{supp}\, u(t) \subset B_{R+t} \quad \text{for all } t \in [0,T]. \tag{3.23}$$

Proof of Theorem 6. In the proof we essentially use the existence – regularity result for the approximate problem (3.13), namely Proposition 1. Since this proposition is valid when g is more regular, i.e., $g_t \in L^2((0,T) \times \mathbb{R}^n)$, the proof is organized in three steps. First we prove that the solution of (3.13) has finite speed of propagation provided that $g_t \in L^2((0,T) \times \mathbb{R}^n)$. Second, passing to the limit as $\varepsilon \to 0$, we prove that the solution of equation (2.1) has finite speed of propagation provided that $g_t \in L^2((0,T) \times \mathbb{R}^n)$. Third, we remove the extra regularity requirement for g using an additional approximation procedure.

Step 1. We apply Proposition 1 to equation (3.13). This gives us the existence and regularity for the solution. We apply (3.17) with test functions of the form $\phi = \alpha u_t^\varepsilon$, where $\alpha \in C_c(\mathbb{R}^{n+1}) \cap W^{1,\infty}(\mathbb{R}^{n+1})$ will be chosen in the sequel. The regularity of αu_t^ε is sufficient in order to use it as a test function. Indeed, for this purpose we use a density lemma which is formulated and proved in Appendix B. Then we get the needed regularity for αu_t^ε as a direct consequence of (3.18), of the Hölder inequality in Orlicz spaces, and the density lemma. So, for any fixed $t \in [0,T]$, we obtain

$$\int_{\mathbb{R}^n} \alpha(s,\cdot)(u_t^\varepsilon)^2(s)\Big|_0^t \tag{3.24}$$
$$= \int_{(0,t)\times\mathbb{R}^n} \alpha_t (u_t^\varepsilon)^2 + \alpha u_t^\varepsilon u_{tt}^\varepsilon - \nabla u^\varepsilon \nabla \alpha u_t^\varepsilon - \alpha \nabla u^\varepsilon \nabla u_t^\varepsilon - \alpha Q_\varepsilon(\cdot, u_t^\varepsilon) u_t^\varepsilon + \alpha g u_t^\varepsilon.$$

Before choosing the functions α, we adapt a classical argument from [25] to evaluate the weighted energy $\frac{1}{2}\int_{\mathbb{R}^n} \alpha(t,\cdot)((u_t^\varepsilon)^2(t) + |\nabla u^\varepsilon(t)|^2$ over a piece of solid cone $K := \{(\tau,x) \in (0,T)\times\mathbb{R}^n : \tau + |x - x_0| \le r\}$, where $r > t$ and $|x_0| > R + r$. Given $\varepsilon > 0$ such that $R + r + \varepsilon < |x_0|$, we also set

$$K_\varepsilon := \{(\tau,x) \in (0,T)\times\mathbb{R}^n : \tau + |x - x_0| \le r + \varepsilon\},$$

so that $K \subset K_\varepsilon \subset D^c$, [2] where $D := \{(\tau,x) \in (0,T)\times\mathbb{R}^n : |x| \le R + \tau\}$ is the light cone of B_R. We define a cut–off function $\sigma \in C^\infty(\mathbb{R})$ such that $\sigma(s) = 1$ for $|s| \le r$, $\sigma(s) = 0$ for $|s| \ge r + \varepsilon$ and $\sigma'(s) \le 0$ for $s \ge 0$. Then we choose $\alpha(\tau,x) = \sigma(\tau + |x - x_0|)$. This choice makes α a function of K_ε. It is easy to verify that α is nonnegative, has the needed regularity and, moreover,

$$\alpha \equiv 1 \quad \text{on } K, \quad \alpha \equiv 0 \quad \text{on } K_\varepsilon^c, \quad \alpha_t + |\nabla\alpha| \le 0 \quad \text{on } (0,T)\times\mathbb{R}^n. \tag{3.25}$$

Next we evaluate the terms in (3.24). Applying $(3.25)_2$ together with

$$B_{r+\varepsilon}(x_0) \cap \operatorname{supp}\, u_1 = \emptyset$$

gives

$$\int_{\mathbb{R}^n} \alpha(\tau,\cdot)(u_t^\varepsilon)^2(\tau)\Big|_0^t = \int_{\mathbb{R}^n} \alpha(t,\cdot)(u_t^\varepsilon)^2(t). \tag{3.26}$$

Integrating by parts with respect to t, we obtain

$$\int_{(0,t)\times\mathbb{R}^n} \alpha u_t^\varepsilon u_{tt}^\varepsilon = \frac{1}{2}\int_{\mathbb{R}^n} \alpha(t,\cdot)(u_t^\varepsilon)^2(t)\,dx - \frac{1}{2}\int_{(0,t)\times\mathbb{R}^n} \alpha_t (u_t^\varepsilon)^2, \tag{3.27}$$

where (3.26) was used to evaluate the boundary term. One more integration by parts gives

$$\int_{(0,t)\times\mathbb{R}^n} \alpha \nabla u^\varepsilon \nabla u_t^\varepsilon = \frac{1}{2}\int_{\mathbb{R}^n} \alpha(t,\cdot)|\nabla u^\varepsilon(t)|^2 - \frac{1}{2}\int_{(0,t)\times\mathbb{R}^n} \alpha_t |\nabla u^\varepsilon|^2, \tag{3.28}$$

[2] Here and in the sequel we denote by S^c the complement of S in $(0,T)\times\mathbb{R}^n$.

where the fact that $B_{r+\varepsilon}(x_0) \cap u_0 = \emptyset$ is used. Moreover, since supp $g \subset D$, then supp $g \cap$ supp $\alpha = \emptyset$, which implies

$$\int_{(0,t)\times\mathbb{R}^n} \alpha g u_t^\varepsilon = 0. \tag{3.29}$$

Inserting the identities (3.26)–(3.29) in (3.24), we get the estimate

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^n} \alpha(t,\cdot)((u_t^\varepsilon)^2(t) + |\nabla u^\varepsilon(t)|^2) \qquad (3.30)\\ &= \frac{1}{2}\int_{(0,t)\times\mathbb{R}^n} \alpha_t((u_t^\varepsilon)^2 + |\nabla u^\varepsilon|^2) - \int_{(0,t)\times\mathbb{R}^n} \nabla u^\varepsilon \nabla \alpha u_t^\varepsilon - \int_{(0,t)\times\mathbb{R}^n} \alpha Q_\varepsilon(\cdot,\cdot,u_t^\varepsilon) u_t^\varepsilon \\ &\le \frac{1}{2}\int_{(0,t)\times\mathbb{R}^n} (\alpha_t + |\nabla\alpha|)((u_t^\varepsilon)^2 + |\nabla u^\varepsilon|^2) \le 0. \end{aligned}$$

Here, to estimate the term $\int_{(0,t)\times\mathbb{R}^n} \nabla u^\varepsilon \nabla\alpha u_t^\varepsilon$ we have applied the Cauchy–Schwarz inequality; to estimate $\int_{(0,t)\times\mathbb{R}^n} \alpha Q_\varepsilon(\cdot,\cdot,u_t^\varepsilon)u_t^\varepsilon$ we use the specific form of the damping term (3.10) and the convexity of Φ_ε; the last estimate of (3.30) follows from $(3.25)_3$.

Since $\alpha \ge 0$, by (3.30) we have $\alpha(t)((u_t^\varepsilon)^2(t) + |\nabla u^\varepsilon(t)|^2) = 0$ a.e. in $\mathbb{R}^n$ and then, using (3.25) we have $(u_t^\varepsilon)^2(t) + |\nabla u^\varepsilon(t)|^2 = 0$ a.e. in $B_{r-t}(x_0)$. Namely, for all $r > t$ and all $|x_0| > R + r$, we obtain $u^\varepsilon(t) =$ Const. for $|x| > R + t$. Then, since $u^\varepsilon(t) \in L^2(\mathbb{R}^n)$, we have

$$\text{supp } u^\varepsilon(t) \subset B_{R+t} \tag{3.31}$$

for all $t \in [0,T]$, which completes the first step in the proof.

Step 2. By (3.2) we have $t^m \le \Phi_\varepsilon(t) + \varepsilon^m$ for $t \ge 0$. Moreover, $\int_{(0,T)\times\mathbb{R}^n} \Phi_\varepsilon(|u_t^\varepsilon|) < \infty$ because of the regularity $(3.18)_2$. Using these estimates together with (3.31), we see that $u_t^\varepsilon \in L^m((0,T)\times\mathbb{R}^n)$. Let us emphasize that this last regularity is a consequence of the finite speed of propagation for u^ε, a fact that was impossible to be obtained directly at the beginning of the proof. Moreover, using this regularity together with by (3.6) and (3.10), we get $Q_\varepsilon(\cdot,u_t^\varepsilon) \in L^{m'}((0,T)\times\mathbb{R}^n)$. Let u be the solution of (2.1) given by Theorem 3. Define $w^\varepsilon = u^\varepsilon - u$, which is a solution of

$$w_{tt}^\varepsilon - \Delta w^\varepsilon + Q_\varepsilon(x,u_t^\varepsilon) - Q(x,u_t) = 0, \tag{3.32}$$

with zero initial data. Then we have enough regularity to write the energy identity for equation (3.32),

$$\tfrac{1}{2}\|w_t^\varepsilon(t)\|_2^2 + \tfrac{1}{2}\|\nabla w^\varepsilon(t)\|_2^2 + \int_{(0,t)\times\mathbb{R}^n} (Q_\varepsilon(\cdot,u_t^\varepsilon) - Q(\cdot,u_t))w_t^\varepsilon = 0. \tag{3.33}$$

Using the monotonicity of the damping term Q in v, we have

$$\int_{(0,t)\times\mathbb{R}^n}(Q_\varepsilon(\cdot,u_t^\varepsilon)-Q(\cdot,u_t))w_t^\varepsilon \ge \int_{(0,t)\times\mathbb{R}^n}(Q_\varepsilon(\cdot,u_t^\varepsilon)-Q(\cdot,u_t^\varepsilon))w_t^\varepsilon. \tag{3.34}$$

The estimate (3.34) allows us to rewrite (3.33) in the form

$$\tfrac{1}{2}\|w_t^\varepsilon(t)\|_2^2+\tfrac{1}{2}\|\nabla w^\varepsilon(t)\|_2^2 \le \int_{(0,t)\times\mathbb{R}^n}|Q_\varepsilon(\cdot,u_t^\varepsilon)-Q(\cdot,u_t^\varepsilon)||w_t^\varepsilon|. \tag{3.35}$$

Now, using Hölder's inequality together with (3.11) and the regularity of $Q_\varepsilon(\cdot,u_t^\varepsilon)$ and $Q(\cdot,u_t^\varepsilon)$, namely that they are in $L^m((0,T)\times\mathbb{R}^n)$, we have

$$I_1 := \int_{(0,T)\times\mathbb{R}^n}|Q_\varepsilon(\cdot,u_t^\varepsilon)-Q(\cdot,u_t^\varepsilon)||w_t^\varepsilon| \le \Big[\int_{A_\varepsilon}|Q_\varepsilon(\cdot,u_t^\varepsilon)-Q(\cdot,u_t^\varepsilon)|^{m'}\Big]^{\frac{1}{m'}}|||w_t^\varepsilon|||_m, \tag{3.36}$$

where $A_\varepsilon=\{(t,x)\in(0,T)\times B_{R+T}:|u_t^\varepsilon|\le\varepsilon\}$. Here the finite speed of propagation for the approximate problem (3.31) was essentially used. By (3.36) and (3.12) together with the assumption (Q7), we have

$$\begin{aligned}
I_1 &\le 2\varepsilon^{m-1}\Big(\int_{A_\varepsilon}|\sigma|^{m'}\Big)^{1/m'}(|||u_t^\varepsilon|||_m+|||u_t|||_m)\\
&\le 2\varepsilon^{m-1}\Big(\int_{(0,T)\times B_{R+T}}|\sigma|^{m'}\Big)^{1/m'}(|||u_t^\varepsilon|||_m+|||u_t|||_m)\\
&\le \text{Const.}\,\varepsilon^{m-1}(|||u_t^\varepsilon|||_m+|||u_t|||_m),
\end{aligned}\tag{3.37}$$

where in the last inequality we have used the L^∞ regularity of σ and the definition of A_ε. From Theorem 4 we know that $|||u_t|||_m<\infty$. It remains to show $\sup_{\varepsilon>0}|||u_t^\varepsilon|||_m<\infty$, which is easily done by using the Hölder inequality (recall that $m<2$) and the finite speed of propagation for (3.13). Thus we have

$$|||u_t^\varepsilon|||_m \le \text{Const.}|||u_t^\varepsilon|||_2. \tag{3.38}$$

From the energy inequality for u^ε

$$\tfrac{1}{2}\|u_t^\varepsilon\|_2^2+\tfrac{1}{2}\|\nabla u^\varepsilon\|_2^2 \le \int_0^t\|g\|_2\|u_t^\varepsilon\|_2.$$

By Gronwall's lemma together with the regularity $g\in L^2((0,T)\times\mathbb{R}^n)$, we get

$$\sup_{t\in[0,T]}\|u_t^\varepsilon(t,\cdot)\|_2 \le \text{Const.},$$

from which it follows that

$$|||u_t^\varepsilon|||_2 \le \text{Const.} \tag{3.39}$$

where the constants in above estimates are independent of ε. Inserting (3.39) and (3.38) in (3.37) gives

$$I_1 \le \text{Const}.\varepsilon^{m-1}.$$

Passing to the limit as $\varepsilon \to 0$ (recall that $m > 1$) we find that $I_1 \to 0$. Then from (3.35) we obtain

$$\tfrac{1}{2}\|w_t^\varepsilon(t)\|_2^2 + \tfrac{1}{2}\|\nabla w^\varepsilon(t)\|_2^2 \to 0, \qquad \text{as } \varepsilon \to 0. \tag{3.40}$$

In view of Cauchy's theorem, and taking into account that $w^\varepsilon(0) = 0$, we derive

$$\|w^\varepsilon(t,\cdot)\|_2 \le \int_0^T \|w_t^\varepsilon\|_2.$$

Together with (3.40) this implies that $w^\varepsilon \to 0$ in $L^2(\mathbb{R}^n)$. Then (up to a subsequence) $u^\varepsilon(t) \to u(t)$ a.e. on $\mathbb{R}^n$. In view of the finite speed of propagation of u^ε this completes the proof of Step 2.

Step 3. When $g_t \notin L^2((0,T)\times\mathbb{R}^n)$, we approximate g with sufficiently regular functions g^ν such that $g^\nu, g_t^\nu \in L^2((0,T)\times\mathbb{R}^n)$, supp $g^\nu(t) \subset B_{R+t+1/\nu}$ for all $t \in [0,T]$, and for which $g^\nu \to g$, as $\nu \to \infty$. By the first part of the proof the solution of the approximate problem

$$\begin{cases} u_{tt}^\nu - \Delta u^\nu + Q(x, u_t^\nu) = g^\nu(t,x), & \text{in } (0,T)\times\mathbb{R}^n, \\ u(0) = u_0, \qquad u_t(0) = u_1, \end{cases} \tag{3.41}$$

satisfies

$$\text{supp } u^\nu(t) \subset B_{R+1/\nu+t}, \qquad \text{for all } t \in [0,T].$$

Now we measure the distance between the solution u of (2.1) and u^ν of equation (3.41). The difference $w^\nu := u - u^\nu$ satisfies the equation

$$\begin{cases} w_{tt}^\nu - \Delta w^\nu + Q(x, u_t^\nu) - Q(x, u_t) = g^\nu(t,x) - g(t,x), & \text{in } (0,T)\times\mathbb{R}^n, \\ u(0) = 0, \qquad u_t(0) = 0. \end{cases} \tag{3.42}$$

Using the energy identity, we have

$$\tfrac{1}{2}\|w^\nu\|_2^2 + \tfrac{1}{2}\|\nabla w^\nu\|_2^2 + \int_{(0,t)\times\mathbb{R}^n} (Q(x,u_t^\nu) - Q(x,u_t))w_t^\nu = \int_{(0,t)\times\mathbb{R}^n} (g^\nu - g)w_t^\nu.$$

By the monotonicity of Q and the Cauchy–Schwarz inequality,

$$\tfrac{1}{2}\|w^\nu\|_2^2 + \tfrac{1}{2}\|\nabla w^\nu\|_2^2 \le \int_0^t \|g^\nu - g\|_2 \|w_t^\nu\|_2.$$

Then from the convergence of g^ν to g in $L^2((0,T)\times\mathbb{R}^n)$ and the Gronwall lemma,

$$\sup_{t\in[0,T]}\|w_t^\nu(t)\|_2 \to 0, \text{as } \nu\to\infty.$$

Finally, since

$$\|w^\nu(t)\|_2 \le \int_0^T \|w_t^\nu\|_2,$$

for all $t\in[0,T]$, we conclude that (up to a subsequence) $u^\nu(t)\to u(t)$ a.e. in $\mathbb{R}^n$. This shows that supp $u(t)\subset B_{R+t}$ for all $t\in[0,T]$, as required. $\square$

4. Existence for nonlinear sources

In this section we prove our main local and global existence results. We start with our general local existence result for the equation (1.1), with damping term Q satisfying the assumptions (Q1)–(Q6) when the growth rate $m\ge 2$, and the assumption (Q7) when $1<m<2$. Moreover, the assumption (Q1) has to be strengthened to

(Q1)$'$ there is $c_0>0$ such that

$$(Q(t,x,v)-Q(t,x,w))(v-w)\ge c_0|v-w|^m$$

for almost all $(t,x)\in(0,T)\times\mathbb{R}^n$ and all $v,w\in\mathbb{R}$.

Clearly, because of the elementary inequality

$$(|v|^{m-2}v-|w|^{m-2}w)(v-w)\ge \text{Const.}|v-w|^m$$

for $m\ge 2$, $v,w\in\mathbb{R}$, the damping term in the model power–like equation, i.e., Q given by (1.2), satisfies the assumption (Q1)$'$.

We consider a nonlinear source f in the form

$$f(x,u)=f_0(x,u)+\mu_2(x)|u|^{q-2}u, \tag{4.1}$$

where $\mu_2\in L^\infty_{\text{loc}}(\mathbb{R}^n)$, $q>1$, and f_0 satisfies the assumption:

(F1) $f_0(x,0)\equiv 0$ and

$$|f_0(x,u_1)-f_0(x,u_2)|\le \mu_1(x)|u_1-u_2|(1+|u_1|^{p-2}+|u_2|^{p-2})$$

for almost all $x\in\mathbb{R}^n$ and all $u_1,u_2\in\mathbb{R}$, where $\mu_1\in L^\infty_{\text{loc}}(\mathbb{R}^n)$ and $p>2$.

Remark 9. It is easy to check that the source term f in the power–like equation (see (1.3)) satisfies (F1), due to the elementary inequality

$$||u_1|^{p-2}u_1-|u_2|^{p-2}u_2|\le \text{Const.}|u_1-u_2|(|u_1|^{p-2}+|u_2|^{p-2}), \qquad u_1,u_2\in\mathbb{R},$$

for $p>2$.

Remark 10. Evidently the essential range for the exponent q is $1 < q < 2$; otherwise for $q \geq 2$ we can combine the second and the first term in (4.1).

Theorem 7. *Let the damping Q satisfy the assumptions* (Q1)$'$ *and* (Q2)*–*(Q6) *when $m \geq 2$, and* (Q7) *when $1 < m < 2$. Let the source f obey* (4.1) *and the assumption* (F1)*. Assume also the growth rates p, m of the nonlinearities satisfy $(p, m) \in I \cup II \cup III$, namely that* (1.5) *holds. Finally let the data be compactly supported and be in the energy space*

$$u_0 \in H^1(\mathbb{R}^n), \qquad u_1 \in L^2(\mathbb{R}^n). \tag{4.2}$$

Then, there is a weak solution of (1.1) *with the properties*

$$u \in C([0,T]; H^1(\mathbb{R}^n)), \quad u_t \in C([0,T]; L^2(\mathbb{R}^n)) \tag{4.3}$$

$$u_t \in L^m((0,T) \times \mathbb{R}^n), \tag{4.4}$$

for $T > 0$ small enough.

Moreover, for exponents $(p, m) \in \overline{I_r} \cup \overline{II_r}$, where

$$\begin{aligned} \overline{I_r} =& \{(p,m) \in \mathbb{R}^2 : 2 < p \leq 1 + r/2, \quad m \geq 2\}, \\ \overline{II_r} =& \{(p,m) \in \mathbb{R}^2 : 2 < p \leq 1 + r/2, \quad 1 < m < 2\} \end{aligned}$$

are the right closure of I and II respectively, and $\mu_2 \equiv 0$, there is a unique weak solution of (1.1) *with the same regularity.*

We emphasize again that the solution which we obtain in Theorem 7 has finite speed of propagation.

Before proving Theorem 7 we need the following:

Proposition 2. *Let u be a function satisfying*

$$\textit{supp } u(t) \subset B_{R+t} \quad \textit{for all } t \in [0,T], \tag{4.5}$$

with smoothness (1.6)*. Let the data u_0 and u_1 have support in B_R and smoothness* (4.2)*. Then the problem*

$$\begin{cases} v_{tt} - \Delta v + Q(t,x,v_t) = f(x,u) & \textit{in } (0,T) \times \mathbb{R}^n, \\ v(0) = u_0, \qquad v_t(0) = u_1, \end{cases} \tag{4.6}$$

has a solution v with regularity (1.6) *and* (1.7) *which satisfies*

$$\textit{supp } v(t) \subset B_{R+t} \quad \textit{for all } t \in [0,T]. \tag{4.7}$$

Proof. This follows some ideas from [6], but uses in a more delicate way the gain of regularity due to the presence of the damping term. Moreover, since we are in $\mathbb{R}^n$ we have to pay attention to the change of the support of functions involved in the approximation procedure. We start by approximating the data with more regular data u_0^ν and u_1^ν in $C_c^\infty(B_R)$, such that

$$u_0^\nu \to u_0 \quad \text{in } H^1(\mathbb{R}^n), \qquad \text{and} \quad u_1^\nu \to u_1 \quad \text{in } L^2(\mathbb{R}^n). \tag{4.8}$$

Next, we approximate the function u in the topology of the energy space with more regular functions $u^\nu \in C_c^\infty([0,T] \times \mathbb{R}^n)$ such that

$$\operatorname{supp} u^\nu(t) \subset B_{R+t+1/\nu} \tag{4.9}$$

for all $t \in [0,T]$. This can easily be done by using standard convolution arguments (see [4]), which of course only inessentiality increases the support.

Then, the existence of a solution v^ν of the problem

$$\begin{cases} v_{tt}^\nu - \Delta v^\nu + Q(t,x,v_t^\nu) = f(x,u^\nu) & \text{in } (0,T)\times\mathbb{R}^n, \\ v^\nu(0) = u_0^\nu, \qquad v_t^\nu(0) = u_1^\nu, \end{cases} \tag{4.10}$$

is assured by Theorem 3. In fact, to apply this result it is only necessary to check that $f(\cdot,u^\nu) \in L^2((0,T)\times\mathbb{R}^n)$. Indeed, using the assumption (Q1) for $u_2 = 0$ and $u_1 = u^\nu$ yields

$$|f(x,u^\nu)| \le \mu_1(x)|u^\nu|(1+|u^\nu|^{p-2}) + |\mu_2(x)||u^\nu|^{q-1}.$$

Now, taking into account the regularity of u^ν, $\mu_1(x)$ and $\mu_2(x)$, there holds $f(\cdot,u^\nu) \in L^\infty((0,T)\times\mathbb{R}^n)$. This together with (4.9) gives us the needed regularity $f(\cdot,u^\nu) \in L^2((0,T)\times\mathbb{R}^n)$, since

$$|||f(\cdot,u^\nu)|||_2 \le |||f(\cdot,u^\nu)|||_\infty \int_{(0,T)\times B_{R+T}} 1.$$

Then, due to Theorem 6 for $1 < m < 2$ and Strauss' theorem [3] for $2 \le m < \infty$, one has

$$\operatorname{supp} v^\nu(t) \subset B_{R+t+1/\nu} \qquad \text{for all } t \in [0,T]. \tag{4.11}$$

The key point in the proof is to show that v^ν is a Cauchy sequence in the topology of the spaces (1.6)–(1.7). This is done in a different way for the three different regions. First of all, the energy identity for $w = v^\nu - v^\mu$ has

[3]Actually for $m \ge 2$, Strauss' argument [25] can be adapted to weak solutions by using the ideas of Steps 1 and 3 in the proof of Theorem 6. In this case Step 2 is no longer needed and can be omitted.

the form

$$\begin{aligned}&\tfrac{1}{2}\|w_t(t)\|_2^2+\tfrac{1}{2}\|\nabla w(t)\|_2^2+\int_{(0,t)\times\mathbb{R}^n}(Q(t,\cdot,u_t^\nu)-Q(t,\cdot,u_t^\mu))w_t\\ =&\tfrac{1}{2}\|w_t(0)\|_2^2+\tfrac{1}{2}\|\nabla w(0)\|_2^2+\int_{(0,t)\times\mathbb{R}^n}(f(\cdot,u^\nu)-f(\cdot,u^\mu))w_t.\end{aligned}\tag{4.12}$$

Let us start with the region I. Using (4.1), (F1) and the Hölder continuity of $|u|^{q-2}u$ (here $1<q<2$) and Remark 10, we obtain the estimate

$$\begin{aligned}&|(f(\cdot,u)-f(\cdot,\bar u),v-\bar v)| \qquad\qquad (4.13)\\ &\le\int_K\big[\|\mu_1\|_{L^\infty(K)}|u-\bar u|(1+|u|^{p-2}+|\bar u|^{p-2})+\|\mu_2\|_{L^\infty(K)}|u-\bar u|^{q-1}\big]|v-\bar v|\\ &\le\text{Const.}\big[\|u-\bar u\|_r(1+\|u\|_{n(p-2)}^{p-2}+\|\bar u\|_{n(p-2)}^{p-2})+\|u-\bar u\|_2^{q-1}\big]\|v-\bar v\|_2\\ &\le\text{Const.}\big[\|\nabla u-\nabla\bar u\|_2(1+\|\nabla u\|_2^{p-2}+\|\nabla\bar u\|_2^{p-2})+\|\nabla u-\nabla\bar u\|_2^{q-1}\big]\|v-\bar v\|_2,\end{aligned}$$

for any $u,\bar u\in H^1(\mathbb{R}^n)$ and $v,\bar v\in L^2(\mathbb{R}^n)$ with support contained in a fixed compact subset K of $\mathbb{R}^n$. The above estimate requires the restriction $n(p-2)\le r$, which is fulfilled in regions I and II since there $p<1+r/2$. Then using (4.12), (Q1)$'$ together with the property supp $u^\nu(t)\subset K=B_{R+T+1}$ (recall that $1<q<2$) and the fact that u^ν is a Cauchy sequence in the energy norm, one arrives at the inequality

$$\begin{aligned}&\tfrac{1}{2}\|w_t(t)\|_2^2+\tfrac{1}{2}\|\nabla w(t)\|_2^2+c_0\int_{(0,t)\times\mathbb{R}^n}|v_t^\nu-v_t^\mu|^m\\ &\quad\le\tfrac{1}{2}\|w_t(0)\|_2^2+\tfrac{1}{2}\|\nabla w(0)\|_2^2+C(T)\int_0^t\|\nabla u^\nu-\nabla u^\mu\|_2^{q-1}\|w_t\|_2.\end{aligned}\tag{4.14}$$

Here and in the sequel we denote by $C(T)$ some T dependent constants, which decrease when $T\to0$. Then, from the Gronwall lemma and the fact that u^ν, u_0^ν and u_1^ν are Cauchy sequences it follows that v^ν is a Cauchy sequence in the energy space, uniformly in time. With the help of the estimate (4.14) once again, together with the knowledge that v^ν is a Cauchy sequence in the energy space, we find that v_t^ν is a Cauchy sequence in the space – time L^m norm.

In the same way, for region II, v^ν is again a Cauchy sequence in the energy norm, uniformly in time. Due to the Hölder inequality in space – time together with (4.11) and the fact that $1<m<2$, we have

$$|||w_t|||_m\le C(T)|||w_t|||_2,\le C(T)\sup_{t\in[0,T]}\|w_t(t,\cdot)\|_2.\tag{4.15}$$

Since v^ν is a Cauchy sequence in the energy norm, uniformly in time, we see that v_t^ν is a Cauchy sequence in the space–time L^m norm.

Region III requires a more careful argument. First we prove that v_t^ν is a Cauchy sequence with respect to the space–time L^m norm, and then show that v^ν is a Cauchy sequence in the energy norm, uniformly in time.

By the energy identity (4.12) and (Q1)$'$ we have

$$|||w_t|||_m^m \le C(T)\Big(\tfrac{1}{2}\|w_t(0)\|_2^2+\tfrac{1}{2}\|\nabla w(0)\|_2^2+\int_{(0,T)\times\mathbb{R}^n}|f(\cdot,u^\nu)-f(\cdot,,u^\mu)||w_t|\Big). \tag{4.16}$$

Given any $u,\bar{u}\in H^1(\mathbb{R}^n)$ and $v,\bar{v}\in L^m(\mathbb{R}^n)$ with support in a fixed compact subset K of $\mathbb{R}^n$, by using (Q1) and Hölder's inequality, we obtain

$$\begin{aligned}|(f(\cdot,u)-f(\cdot,\bar{u}),v-\bar{v})| &\le \|\mu_1\|_{L^\infty(K)}\int_K|u-\bar{u}|(1+|u|^{p-2}+|\bar{u}|^{p-2})|v-\bar{v}|\\ &\quad+\|\mu_2\|_{L^\infty(K)}\int_K|u-\bar{u}|^{q-1}|v-\bar{v}| \qquad (4.17)\\ &\le \text{Const.}\Big[\|u-\bar{u}\|_r(1+\|u\|_{s(p-2)}^{p-2}+\|\bar{u}\|_{s(p-2)}^{p-2})+\|u-\bar{u}\|_{m'(q-1)}^{q-1}\Big]\|v-\bar{v}\|_m,\end{aligned}$$

where $\frac{1}{s}+\frac{1}{m}+\frac{1}{r}=1$. This is possible because $m\ge r/(r+1-p)$ and $p>2$. Now, to estimate the $L^{s(p-2)}$ norm with the energy norm we need the restriction $s(p-2)\le r$ which again is exactly $m\ge r/(r+1-p)$. By the Sobolev embedding theorem (recall that $m'(q-1)<2$),

$$\begin{aligned}&|(f(\cdot,u)-f(\cdot,\bar{u}),v-\bar{v})| \qquad (4.18)\\ &\le\text{Const.}\big[\|\nabla u-\nabla\bar{u}\|_2(1+\|\nabla u\|_2^{p-2}+\|\nabla\bar{u}\|_2^{p-2})+\|\nabla u-\nabla\bar{u}\|_2^{q-1}\big]\|v-\bar{v}\|_m.\end{aligned}$$

Then using (4.18), the Hölder inequality in time, the facts that u^ν is a Cauchy sequence and that supp $u^\nu(t)\subset K:=B_{R+T+1}$ gives

$$\int_{(0,T)\times\mathbb{R}^n}|f(\cdot,u^\nu)-f(\cdot,,u^\mu)||w_t|\le C(T)\|\nabla u^\nu-\nabla u^\mu\|_{L^\infty(0,T;L^2(\mathbb{R}^n))}^{q-1}|||w_t|||_m. \tag{4.19}$$

Inserting (4.19) in (4.16) yields

$$\begin{aligned}|||w_t|||_m^m\le C(T)\Big(&\tfrac{1}{2}\|w_t(0)\|_2^2+\tfrac{1}{2}\|\nabla w(0)\|_2^2 \qquad (4.20)\\ &+\|\nabla u^\nu-\nabla u^\mu\|_{L^\infty(0,T;L^2(\mathbb{R}^n))}^{q-1}|||w_t|||_m\Big).\end{aligned}$$

Now we apply the elementary observation that if, for given nonnegative sequences x_ν, h_ν and k_ν, the inequality

$$x_\nu^m\le k_\nu+h_\nu x_\nu, \tag{4.21}$$

is fulfilled and $h_\nu, k_\nu \to 0$, then $x_\nu \to 0$. By (4.20) and (4.21), since u^ν and the data are Cauchy sequences with respect to the energy norm, it follows that v_t^ν is a Cauchy sequence in the L^m space–time norm. Next we prove that v^ν is a Cauchy sequence with respect to the energy norm, uniformly in time. Substituting (4.19) into the energy identity (4.12), and taking into account that the integral in the left hand side is nonnegative due to (Q1)$'$, we arrive at

$$\begin{aligned}&\tfrac{1}{2}\|w_t(t)\|_2^2 + \tfrac{1}{2}\|\nabla w(t)\|_2^2\\ &\le \tfrac{1}{2}\|w_t(0)\|_2^2 + \tfrac{1}{2}\|\nabla w(0)\|_2^2 + C(T)\|\nabla u^\nu - \nabla u^\mu\|_{L^\infty(0,T;L^2(\mathbb{R}^n))}^{q-1} |||w_t|||_m.\end{aligned}$$

Then since all the sequences on the right hand side are Cauchy sequences, it follows that v^ν is a Cauchy sequence in the energy norm, uniformly in time. Thus, in all three regions v^ν is a Cauchy sequence in the topology of the spaces (1.6)–(1.7).

For all three regions, let us denote by v the limit of the sequence v^ν. It is easy to show that v is a unique weak solution of equation (4.6) in the sense of [16] and that this solution satisfies the energy identity. Finally from (4.11) we get (4.7). This completes the proof. □

Proof of Theorem 7. The argument is organized as follows. For all three regions (see Figure 1) we apply the Schauder fixed point theorem. Further it will be shown that, when $\mu_2 \equiv 0$ in the region $\overline{\mathrm{I}_r} \cup \overline{\mathrm{II}_r}$, the result can be improved by applying the contraction mapping principle instead of the Schauder theorem.

For any u with the regularity (1.6) and satisfying (4.5), we define the map $v = \Phi(u)$, where v is the solution of equation (4.6). Set

$$\begin{aligned}X_T =&\{u \in C([0,T]; H^1(\mathbb{R}^n)) \cap C^1([0,T]; L^2(\mathbb{R}^n)) :\\ &\quad \operatorname{supp} u(t) \subset B_{R+t} \text{ for all } t \in [0,T],\ u(0) = u_0, u_t(0) = u_1\}.\end{aligned}$$

From Proposition 2, we see that the map Φ is well defined from X_T into itself.

Further, it will be shown that for sufficiently large R_0 and for sufficiently small T, Φ maps the ball B_{R_0} of radius R_0 in X_T into itself.

In the region I∪II we directly estimate the $H^1 \times L^2$ norms in the following way (actually this estimate is fulfilled in a larger region $\overline{\mathrm{I}_r} \cup \overline{\mathrm{II}_r}$). Applying (4.18) for $\bar{u} = \bar{v} = 0$ and using the first part of assumption (F1) gives

$$|(f(\cdot, u(t)), v_t(t))| \le C(T)(1 + \|\nabla u(t)\|_2^{p-1})\|v_t(t)\|_2. \tag{4.22}$$

In the last estimate the elementary inequality $x + x^\alpha \le \text{Const.}(1 + x^\alpha)$, for $x \ge 0$ and $\alpha > 1$, was applied, together with the fact that $p > 2$. Then using

(4.22), the energy identity for v, and the assumptions (Q1)$'$ (for the region I) or (Q7) (for the region II) there results

$$\tfrac{1}{2}\|v_t(t)\|_2^2 + \tfrac{1}{2}\|\nabla v(t)\|_2^2 \le \tfrac{1}{2}\|u_1\|_2^2 + \tfrac{1}{2}\|\nabla u_0\|_2^2 + \text{Const.}\int_0^t (1 + R_0^{p-1})\|v_t\|_2, \tag{4.23}$$

where we take into account that $u \in B_{R_0}$. Applying the Gronwall lemma and choosing T sufficiently small and R_0 in a convenient way, completes this part of the proof (recall that $C(T)$ decreases when $T \to 0$).

For region III, the $H^1 \times L^2$ estimate is derived by first passing through a space – time L^m estimate. To prove this estimate, put $\bar{u} = \bar{v} = 0$ in (4.18), so that

$$\begin{aligned} |(f(\cdot, u(t)), v_t(t))| \le& C(T)(1 + \|\nabla u(t)\|_2^{p-2})\|\nabla u(t)\|_2\|v_t(t)\|_m \\ \le& \text{Const.}\, {R_0}^{p-1}\|v_t(t)\|_m, \end{aligned} \tag{4.24}$$

where $u \in B_{R_0}$ and $R_0 \ge 1$ without loss of generality. Further, using the energy identity for v together with (4.24) and the assumption (Q3), we have

$$|||v_t|||_m^m \le C(T)(1 + {R_0}^{p-1}T^{1/m'}|||v_t.|||_m).$$

In the last estimate the Hölder inequality has been applied in time. This yields

$$|||v_t|||_m \le C(T)(1 + R_0^{(p-1)/(m-1)}T^{1/m}). \tag{4.25}$$

This is the required space–time L^m estimate.

Now to obtain an $H^1 \times L^2$ estimate, we use (4.25) and the energy identity together with Hölder's inequality in time, to get

$$\tfrac{1}{2}\|v_t(t)\|_2^2 + \tfrac{1}{2}\|\nabla v(t)\|_2^2 \le C(T)(1 + {R_0}^{p-1}T^{1/m'} + {R_0}^{(p-1)m'}T). \tag{4.26}$$

This completes the L^2 estimate, for sufficiently small T and conveniently chosen R_0.

Now a final choice can be made for R_0 and T for all three regions I, II and III: these fixed numbers R_0 and T will be used in the sequel. First we prove that Φ is a compact map in B_{R_0} for appropriate R_0. In the regions I and II we refine the estimate (4.13) using a conveniently chosen L^{r_0} norm, where r_0 $r_0 \in (p, r)$ and $p \le 1 + r_0/2$, thus

$$\begin{aligned} &|(f(\cdot, u(t)) - f(\cdot, \bar{u}(t)), v_t(t) - \bar{v}_t(t))| \\ &\le \text{Const.}\Big[\|u(t) - \bar{u}(t)\|_{r_0}(1 + \|u(t)\|_{s(p-2)}^{p-2} \\ &+ \|\bar{u}(t)\|_{s(p-2)}^{p-2}) + \|u(t) - \bar{u}(t)\|_{r_0}^{q-1}\Big]\|v_t(t) - \bar{v}_t(t)\|_2, \end{aligned} \tag{4.27}$$

where $\frac{1}{2}+\frac{1}{r_0}+\frac{1}{s}=1$. Since $p\le 1+r_0/2$, which is equivalent to $s(p-2)\le r_0<r$, we can use the definition of X_T and rewrite (4.27), (keeping the essential part $\|u(t)-\bar{u}(t)\|_{r_0}$),

$$\begin{aligned}&|(f(\cdot,u(t))-f(\cdot,\bar{u}(t)),v_t(t)-\bar{v}_t(t))|\\&\le C(T)\Big[\|u(t)-\bar{u}(t)\|_{r_0}(1+\|\nabla u(t)\|_2^{p-2}\\&+\|\nabla\bar{u}(t)\|_2^{p-2})+\|u(t)-\bar{u}(t)\|_{r_0}^{q-1}\Big]\|v_t(t)-\bar{v}_t(t)\|_2.\end{aligned}\tag{4.28}$$

Now let $u^\nu\in B_{R_0}$ and set $v^\nu=\Phi(u^\nu)$. Combining the energy identity with (4.28) for $u=u^\nu$, $\bar{u}=u^\mu$, there results

$$\|v^\nu-v^\mu\|_{X_T}^2\le C(T)(1+R_0^{p-2})\int_0^T\|u^\nu-u^\mu\|_{r_0}^{q-1}\|v^\nu-v^\mu\|_{X_T}.\tag{4.29}$$

Here we have used the fact that u^ν is bounded in X_T because $u^\nu\in B_{R_0}$, and also that $q<2$. Since u^ν is a bounded sequence in $C([0,T];H^1(B_{R+T+1}))$ $\cap C^1([0,T];L^2(B_{R+T+1}))$, then by the compactness argument given in Lemma 2, which is postponed to the end of this section, one finds that, up to a subsequence, u^ν is a Cauchy sequence in $C([0,T];L^{r_0}(B_{R+T+1}))$; recall here that $2<p<r_0<r$. This completes the proof of compactness for the map Φ in the region I$\,\cup\,$II.

For region III the proof is more delicate. We start with a slight modification of (4.17), using the L^{r_0} norm with $r_0\in(p,r)$ chosen such that

$$m\ge\frac{r_0}{r_0-p+1}>\frac{r}{r-p+1}.$$

Then

$$\begin{aligned}&|(f(\cdot,u(t))-f(\cdot,\bar{u}(t)),v_t(t)-\bar{v}_t(t))|\\&\le C(T)\Big[\|u(t)-\bar{u}(t)\|_{r_0}(1+\|u(t)\|_{s(p-2)}^{p-2}\\&+\|\bar{u}(t)\|_{s(p-2)}^{p-2})+\|u(t)-\bar{u}(t)\|_{r_0}^{q-1}\Big]\|v_t(t)-\bar{v}_t(t)\|_m,\end{aligned}\tag{4.30}$$

where $\frac{1}{m}+\frac{1}{r_0}+\frac{1}{s}=1$. The $L^s(p-2)$ norm can be controlled by the energy norm because $s(p-2)<r$ due to the choice of r_0. Then, using the estimate (4.17) and arguments similar to those for region I$\,\cup\,$II, one derives

$$\|v^\nu-v^\mu\|_{X_T}^2\le C(T)(1+R_0^{p-2})\int_0^T\|u^\nu-u^\mu\|_{r_0}^{q-1}\|v_t^\nu-v_t^\mu\|_m.\tag{4.31}$$

Next using Hölder's inequality in time, we get

$$\|v^\nu - v^\mu\|_{X_T}^2 \le C(T)(1+R_0^{p-2})\Big[\int_0^T \|u^\nu - u^\mu\|_{r_0}^{m'(q-1)}\Big]^{1/m'} \|\|v_t^\nu - v_t^\mu\|\|_m. \quad (4.32)$$

This estimate, together with the bound on the L^m space–time norm given by (4.25), and the compactness Lemma 2 below, completes the proof of the compactness of the map Φ in region III. Application of the Schauder fixed point theorem then complete the proof of the local existence result.

This result can be improved in the region $\overline{\mathrm{I}_r} \cup \overline{\mathrm{II}_r}$ when $\mu_2 \equiv 0$ (which is equivalent to $q \ge 2$). In fact in this region the map Φ is actually a contraction map, for one has the estimate

$$\begin{aligned}\|v - \bar v\|_{X_T}^2 \le &\text{Const.} \int_0^T \|\nabla u - \nabla \bar u\|_2 (1 + \|\nabla u\|_2^{p-2} + \|\nabla \bar u\|_2^{p-2}) \|v_t - \bar v_t\|_2 \\ \le &C(T)(1 + R_0^{p-2}) T \|\nabla u - \nabla \bar u\|_{L^\infty(0,T;L^2(\mathbb{R}^n))} \|v - \bar v\|_{X_T}.\end{aligned}$$

This completes the proof of existence–uniqueness result for this region. □

Finally, we formulate a global existence result when $p \le m$. This requires a further assumption for the antiderivative of the first part of the source term, $F_0(x,u) = \int_0^u f_0(x,\eta)d\eta$, namely

(F2) $F_0(x,u) \ge \mu_1(x)(|u|^p - c_6)$, where c_6 is a constant.

Theorem 8. *Let the damping Q satisfy assumptions* (Q1)$'$ *and* (Q2)–(Q6) *when $m \ge 2$, and* (Q7) *when $1 < m < 2$. Suppose the source f satisfies* (4.1) *and the assumptions* (F1)–(F2)*, and assume $2 < p < r$, $1 < q < p \le m$. Then for any compactly supported data in the energy space $u_0 \in H^1(\mathbb{R}^n)$, $u_1 \in L^2(\mathbb{R}^n)$, the Cauchy problem* (1.1) *has a global solution with the regularity* (1.6)–(1.7)*.*

Proof. From Theorem 7, we know that the Cauchy problem (1.1) has a local solution. Using the finite propagation speed one can prove a continuation principle as in [23]. Denote by $T_{\max}$ the life–span of the solution: we shall show that $T_{\max} = \infty$. The proof is done by contradiction. Assuming that $T_{\max} < \infty$, from the continuation principle it follows that

$$\lim_{t \to T_{\max}^-} \left(\tfrac12 \|u_t(t)\|_2^2 + \tfrac12 \|\nabla u(t)\|_2^2\right) = \infty. \quad (4.33)$$

We construct the functional

$$\mathcal{I}(t) = \tfrac12 \|u_t(t)\|_2^2 + \tfrac12 \|\nabla u(t)\|_2^2 + \int_{\mathbb{R}^n} F_0(x,u)\,dx + \tfrac1q \int_{\mathbb{R}^n} |\mu_2(x)||u|^q + A, \quad (4.34)$$

where A is a conveniently chosen (large) constant, such that

$$\int_{\mathbb{R}^n} F_0(x,u)\,dx + A \geq \int_{\mathbb{R}^n} \mu_1(x)|u|^p \geq 0. \tag{4.35}$$

This is possible because of the assumption (F2). Indeed

$$\begin{aligned}\int_{\mathbb{R}^n} F_0(x,u)\,dx &= \int_{B_{R+T_{\max}}} F_0(x,u)\,dx \geq \int_{B_{R+T_{\max}}} \mu_1(x)(|u|^p - c_6) \quad (4.36)\\ &\geq \int_{B_{R+T_{\max}}} \mu_1(x)|u|^p - c_6\|\mu_1\|_{L^\infty(B_{R+T_{\max}})}(R+T_{\max})^n\omega_n,\end{aligned}$$

where ω_n is the measure of the unit ball B_1 in $\mathbb{R}^n$. Here we have used the finite speed of propagation and the regularity of μ_1. Now the choice of the constant A is evident: $A \geq c_6\|\mu_1\|_{L^\infty(B_{R+T_{\max}})}(R+T_{\max})^n\omega_n$.

We shall show that

$$\mathcal{I}(t)' \leq \text{Const.}(\mathcal{I}(t)+1), \tag{4.37}$$

which by Gronwall's lemma shows that $\mathcal{I}$ has exponential growth. Together with the choice of the constant A this gives a contradiction.

We rewrite $\mathcal{I}(t)$ in the form

$$\mathcal{I}(t) = E(t) + 2\int_{\mathbb{R}^n} F_0(x,u) + \frac{2}{q}\int_{\mathbb{R}^n} \mu_2^+(x)|u|^q + A, \tag{4.38}$$

where E is the energy of the solution, namely

$$E(t) = \tfrac{1}{2}\|u_t(t)\|_2^2 + \tfrac{1}{2}\|\nabla u(t)\|_2^2 - \int_{\mathbb{R}^n} F_0(x,u)\,dx - \tfrac{1}{q}\int_{\mathbb{R}^n} \mu_2(x)|u|^q$$

and $\mu_2^+(x) = (|\mu_2(x)| + \mu_2(x))/2$.

From (4.38) one can calculate $\mathcal{I}'$, namely

$$\begin{aligned}\mathcal{I}'(t) &= -\int_{\mathbb{R}^n} Q(t,x,u_t)u_t + 2\int_{\mathbb{R}^n} f_0(x,u)u_t + 2\int_{\mathbb{R}^n} \mu_2^+(x)|u|^{q-2}uu_t\\ &= I_1 + I_2 + I_3. \end{aligned}\tag{4.39}$$

We estimate each term on the right hand side of the above identity. The main estimate for I_1 follows immediately from assumption (3), namely

$$I_1 \leq -c_1\|u_t\|_m^m. \tag{4.40}$$

For I_2, using (F1) and the finite speed of propagation, we have

$$|I_2|/2 \leq \int_{B_{R+t}} \mu_1(x)|u||u_t| + \int_{B_{R+t}} \mu_1(x)|u|^{p-1}|u_t|. \tag{4.41}$$

The first integral in the above inequality is easily estimated (using the fact that the constant in the Poincarè inequality in a ball is proportional to the radius of the ball) as follows:

$$\int_{B_{R+t}} \mu_1(x)|u||u_t| \leq \|\mu_1\|_{L^\infty(B_{R+T_{\max}})} \left[(R+T_{\max})^2\|\nabla u\|_2^2 + \|u_t\|_2^2\right] \tag{4.42}$$
$$\leq \text{Const.}\mathcal{I}(t).$$

For the second integral in (4.41) one has

$$\int_{B_{R+t}} \mu_1(x)|u|^{p-1}|u_t|dx \leq C(\varepsilon)\int_{\mathbb{R}^n} \mu_1(x)|u|^p + \varepsilon p^{-1}\|\mu_1\|_{L^\infty(B_{R+T_{\max}})}\|u_t\|_p^p \tag{4.43}$$

$$\leq C(\varepsilon)\int_{\mathbb{R}^n} \mu_1(x)|u|^p + \text{Const.}\varepsilon(\|u_t\|_2^2 + \|u_t\|_m^m) \leq C(\varepsilon)\mathcal{I}(t) + \text{Const.}\varepsilon\|u_t\|_m^m.$$

Here one uses the Young inequality, the convexity of the function u^y for $y > 2$ and $u \geq 0$, and the fact that $2 < p < m$. We also used the definition (4.34) of $\mathcal{I}$ and the property (4.35), which follows from the choice of A. To estimate the last term I_3 in (4.39) one has

$$I_3 \leq C(\varepsilon)\int_{B_{R+T_{\max}}} |\mu_2(x)||u|^{(q-1)m'} + \varepsilon\|\mu_2\|_{L^\infty(B_{R+T_{\max}})}\|u_t\|_m^m \tag{4.44}$$
$$\leq C(\varepsilon)\Big(\int_{\mathbb{R}^n} |\mu_2(x)||u|^q\Big)^{m'q'} \|\mu_2\|_{L^\infty(B_{R+T_{\max}})}^{1-m'/q'}[(R+T_{\max})^n\omega_n]^{1-m'/q'}$$
$$+ \varepsilon\text{Const.}\|u_t\|_m^m$$
$$\leq C_1(\varepsilon)\Big(\int_{\mathbb{R}^n} |\mu_2(x)||u|^q\Big)^{m'/q'} + \varepsilon\text{Const.}\|u_t\|_m^m$$
$$\leq C_1(\varepsilon)\mathcal{I}(t)^{m'/q'} + \varepsilon\text{Const.}\|u_t\|_m^m.$$

Here and in the sequel $C_1(\varepsilon)$ denote constants which increase when $\varepsilon \to 0$. Moreover, the Young inequality is used twice, (4.34), (4.35), together with the fact that $q' > m'$ (since $q < m$).

Inserting the estimates (4.40)–(4.44) in (4.39) gives

$$\mathcal{I}'(t) \leq C_1(\varepsilon)(\mathcal{I}(t) + \mathcal{I}^{m'/q'}(t)) + \text{Const.}\varepsilon\|u_t\|_m^m - c_1\|u_t\|_m^m,$$

where c_1 is the constant appearing in assumption (Q2) when $T = T_{\max}$. Then, for sufficiently small ε, we have

$$\mathcal{I}'(t) \leq \text{Const.}(\mathcal{I}(t) + \mathcal{I}^{m'/q'}(t)) \leq \text{Const.}(\mathcal{I}(t) + 1),$$

where in the last estimate we have used the elementary inequality $x^\alpha \leq 1+x$, for $x \geq 0$ and $0 \leq \alpha \leq 1$. This completes the proof. $\square$

We can now give the

Proof of Corollary 1. We know from Theorem 7 that (1.8) has a local solution u with regularity (1.6) and (1.7). Assume for contradiction that the life–span $T_{\max}$ of the solution is finite. Then, since

$$\frac{1}{(1+T_{\max})^{\beta}} \le \alpha(t) \le 1,$$

all the assumptions (Q1)$'$, (Q2) –(Q6) (recall that $m \ge 2$) are fulfilled for $T = T_{\max}$. From Theorem 8 we get a contradiction, which complete the proof. □

We finish the section with the proof of the following compactness lemma, used above in the proof of local existence.

Lemma 2. *Let $\Omega \subset \mathbb{R}^n$ be bounded and $2 < r_0 < r$, where $r = 2n/(n-2)$ if $n \ge 3$, $r = \infty$ if $n = 1, 2$. Then the embedding*

$$\mathcal{K}_1 := C([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \hookrightarrow C([0,T]; L^{r_0}(\Omega))$$

is compact.

Proof. Let u^n be a sequence in $\mathcal{K}_1$ such that

$$\sup_{t\in[0,T]} \|u^n(t)\|_{H^1(\Omega)} \le \text{Const.} \qquad \text{and} \qquad \sup_{t\in[0,T]} \|u_t^n(t)\|_2 \le \text{Const.} \tag{4.45}$$

for all $n \in \mathbb{N}$. By the interpolation inequality (see [4]) for $t, s \in [0,T]$ we have

$$\|u^n(t) - u^n(s)\|_{r_0} \le \|u^n(t) - u^n(s)\|_2^{\beta} \|u(t) - u(s)\|_r^{1-\beta}, \tag{4.46}$$

where $\frac{1}{r_0} = \frac{\beta}{2} + \frac{1-\beta}{r}$. Moreover, by $(4.45)_2$ and Cauchy's theorem

$$\|u^n(t) - u^n(s)\|_2 \le \left| \int_s^t \|u_t^n(\eta)\|_2 \, d\eta \right| \le \text{Const.}|t-s|. \tag{4.47}$$

Using (4.45)–(4.47) and Sobolev embedding we get

$$\|u^n(t) - u^n(s)\|_{r_0} \le \text{Const.}|t-s|^{\beta}.$$

Hence, u^n is an equicontinuous sequence in $C([0,T]; L^{r_0}(\Omega))$. Moreover, by Rellich's Theorem (see [4]), for each $t \in [0,T]$ the sequence $u^n(t)$ has a convergent subsequence in $L^{r_0}(\Omega)$. Together with Ascoli's Theorem (see for example [22]), this completes the proof. □

Appendix A. Proof of Proposition 1

We first recall additional arguments of Orlicz space theory (see [1] and [21]). The N–function Φ is said to satisfy the Δ_2 condition, or to be Δ_2–regular, if there is a positive constant $k > 0$ such that $\Phi(2s) \le k\Phi(s)$ for $s \ge 0$. An important example of Δ_2–regular function is the function Φ_ε constructed in Lemma 1.

The norm $\|u\|_\Phi$ in the Orlicz space $L^\Phi(A)$, where $A = \mathbb{R}^n$ or $A = (0,T) \times \mathbb{R}^n$, can be defined by

$$\|u\|_\Phi = \inf\left\{k > 0 : \textstyle\int_A \Phi(|u|/k) \le 1\right\} = \inf\left\{\tfrac{1}{k}(1 + \textstyle\int_A \Phi(k|u|)) : k > 0\right\}. \tag{A.1}$$

Moreover,

$$C_c^\infty(A) \text{ is dense in } L^\Phi(A) \text{ and in } L^\Phi(A) \cap H_0^1(A). \tag{A.2}$$

Next, the complementary function Ψ is Δ_2–regular if and only if there is a constant $l > 1$ such that Φ satisfies the inequality

$$2l\Phi(s) \le \Phi(ls) \qquad \text{for all } s \ge 0.$$

A direct consequence of this fact together with the property (3.5) of the function Φ_ε is that the complementary function Ψ_ε is Δ_2–regular.

Let (Φ, Ψ) be a complementary pair of N functions, both satisfying the Δ_2 condition. Then

$$[L^\Phi(A)]^* = L^\Psi(A), \qquad [L^\Psi(A)]^* = L^\Phi(A), \tag{A.3}$$

where X^* is the topological dual space of X. In addition, given a sequence u_n in $L^\Phi(A)$, the following properties hold

$$u_n \text{ is bounded in } L^\Phi(A) \text{ if and only if the sequence } \textstyle\int_A \Phi(|u_n|) \text{ is bounded}, \tag{A.4}$$

$$u_n \to u \text{ in } L^\Phi(A) \text{ if and only if } \textstyle\int_A \Phi(|u_n - u|) \to 0, \tag{A.5}$$

and

$$\textstyle\int_A \Phi(|u_n|) \to \int_A \Phi(|u|) \text{ and } u_n \to u \text{ a.e. in } A \text{ implies } u_n \to u \text{ in } L^\Phi(A). \tag{A.6}$$

Clearly,

$$\Psi'(t) = \inf\{s > 0 : \Phi'(s) > t\}. \tag{A.7}$$

Moreover, the following continuous embedding holds

$$L^\Phi((0,T) \times \mathbb{R}^n) \hookrightarrow L^1(0,T; L^\Phi(\mathbb{R}^n)). \tag{A.8}$$

We need two lemmas, which are generalizations of well–known facts about Lebesgue spaces to Orlicz spaces. We sketch the proof of these lemmas to keep the paper self–contained and because we have not a precise reference.

Lemma 3. *Suppose that Φ is a Δ_2–regular N–function and $A \subset \mathbb{R}^k$. If $u_n \to u$ in $L^\Phi(A)$, then there is $h \in L^\Phi(A)$ such that (up to a subsequence) $u_n \to u$ a.e. in A as $n \to \infty$ and $|u_n| \leq h$ a.e. in A for all $n \in \mathbb{N}$.*

Proof. This a slight modification of the proof of Theorem IV.9 of [4]. We use the monotonicity and continuity of Φ, which are direct consequences of the definition of N–functions, together with the convergence properties (A.4) and (A.5). □

The second lemma is the extension to Orlicz spaces of the continuity of Nemitskii operators in Lebesgue spaces (see [2]).

Lemma 4. *Suppose that Φ and Ψ are Δ_2–regular N–functions and $A \subset \mathbb{R}^k$. Suppose moreover that $F : A \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that*

$$\Psi(|F(x,s)|) \leq \text{Const.}\Phi(|s|) \tag{A.9}$$

for $s \geq 0$. Then the Nemitskii operator $u \to F(\cdot,u)$ is continuous from $L^\Phi(A)$ to $L^\Psi(A)$.

Proof. From the definition of the Orlicz space and the assumption (A.9) we have that the operator $u \to F(\cdot,u)$ is well defined from $L^\Phi(A)$ to $L^\Psi(A)$. Applying Lemma 3 for a convergent sequence $u_n \to u$ in $L^\Phi(A)$, together with the continuity of F in the second variable, we have that (up to a subsequence)

$$F(\cdot,u_n) \to F(\cdot,u) \qquad \text{a.e. in } A, \tag{A.10}$$

and

$$|u_n| \leq h \qquad \text{a.e. in } A, \tag{A.11}$$

for all $n \in \mathbb{N}$, where $h \in L^\Phi(A)$. Since the continuity of Ψ, as an N–function, we have $\Psi(|F(\cdot,u_n)| \to \Psi(|F(\cdot,u)|$ a.e. in A. Using estimates (A.9) and (A.11) we have

$$\Psi(|F(\cdot,u_n)|) \leq \text{Const.}\Phi(|u_n|) \leq \text{Const.}\Phi(h).$$

Now $\Phi(h) \in L^1(A)$ by the definition of Orlicz space. Hence, by the Lebesgue Dominated Convergence Theorem,

$$\int_\Omega \Psi(|F(\cdot,u_n)|) \to \int_\Omega \Psi(|F(\cdot,u)|)$$

and then using (A.10) and the convergence property (A.6), we derive $F(\cdot,u_n) \to F(\cdot,u)$ in $L^\Psi(A)$ (up to a subsequence). Since this is true for any sequence $u_n \to u$, this completes the proof. □

Proof of Proposition 1. We follow the arguments of Theorem 3 and Theorem 4, using convenient modifications. The first part of the proof, concerning the existence of the regular solution u^ρ in the ball B_ρ, is not essentially affected, since the argument requires only the use of Lebesgue spaces. The only difference in fact is that, since $1 < m < 2$, we do not need to approximate $u_{1\rho}$ with u_{1n} in $L^{2(m-1)}(B_\rho)$. The convergence $u_{1n} \to u_{1\rho}$ in $L^2(B_\rho)$, which is a consequence of convergence in $H_0^1(B_\rho)$, implies that $|u_{1n}|^{m-1} \to |u_{1\rho}|^{m-1}$ in $L^2(B_\rho)$. Thus, the estimate (2.19) can be rewritten

$$\|u_n''(0)\|_2 \le \|\Delta u_{0n}\|_2 + \|g_\rho(0)\|_2 + \text{Const.}\||u_{1n}|^{m-1}\|_2. \tag{A.12}$$

Due to the fact that Q_ε is time independent, together with $Q_\varepsilon(x,v)v \ge 0$ and $(Q_\varepsilon)_v(x,v) \ge 0$ (because of the strict convexity of Φ_ε and (3.1)), we can use the first and second energy identities. This yields the main estimate (2.20) in the simplified form,

$$\sup_{t\in(0,T)} \|u_n''(t)\|_2^2 + \|\nabla u_n'(t)\|_2^2 \le \text{Const.}\left(\||(g_\rho)_t\||_2^2 + \|u_n''(0)\|_2^2 + \|\nabla u_{1n}\|_2^2\right). \tag{A.13}$$

That is, the function u^ρ has the regularity described in Theorem 4.

Until now, working in the ball B_ρ, no essential difference with Theorem 4 arises. On the other hand, a difference is manifested when we pass to the limit as $\rho \to \infty$, which requires using the Orlicz space $L^{\Phi_\varepsilon}((0,T)\times\mathbb{R}^n)$ instead of the Lebesgue space $L^m((0,T)\times\mathbb{R}^n)$. Indeed from the estimate (2.7), one cannot derive the boundedness of u_t^ρ in $L^m((0,T)\times\mathbb{R}^n)$. Instead, using the convexity of Φ_ε and the fact that $\Phi_\varepsilon(0) = 0$, we have

$$\Phi_\varepsilon(t) = \int_0^t \Phi_\varepsilon'(\eta)\,d\eta \le \Phi_\varepsilon'(t)t \qquad \text{for all } t \ge 0.$$

Thus, by the particular form (3.10) of Q_ε and the previous estimate, we derive

$$\int_{(0,T)\times\mathbb{R}^n} \Phi_\varepsilon(|u_t^\rho|) \le \text{Const.}\int_{(0,T)\times\mathbb{R}^n} Q_\varepsilon(\cdot, u_t^\rho)u_t^\rho.$$

Further, the boundedness of u_t^ρ in the Orlicz space $L^{\Phi_\varepsilon}((0,T)\times\mathbb{R}^n)$ follows from the previous estimate and the property (A.4). Also by the continuity of Φ_ε', the strict convexity of the function Φ_ε and (A.7), there arises $\Psi_\varepsilon'(t) = (\Phi_\varepsilon')^{-1}(t)$, where Ψ_ε is the complementary function of Φ_ε. Then, using the properties (3.1) and (3.7) of Φ_ε, there holds, for $t \ge 0$,

$$\Psi_\varepsilon(\Phi_\varepsilon'(t)) = \int_0^t (\Psi_\varepsilon\circ\Phi_\varepsilon')'(\eta)\,d\eta = \int_0^t \Phi_\varepsilon''(\eta)\eta\,d\eta \le c_{2\varepsilon}\int_0^t \Phi_\varepsilon'(\eta)\,d\eta = c_{2\varepsilon}\Phi_\varepsilon(t). \tag{A.14}$$

Next by the monotonicity and the Δ_2–regularity of Ψ_ε, together with (A.14).

$$\Psi_\varepsilon(|Q_\varepsilon(x,v)|) \le \Psi_\varepsilon(\|\sigma\|_\infty \Phi'_\varepsilon(|v|) \le c_{3\varepsilon}\Psi_\varepsilon(\Phi'_\varepsilon(|v|)) \le c_{4\varepsilon}\Phi_\varepsilon(|v|) \qquad \text{(A.15)}$$

for some constants $c_{3\varepsilon}$, $c_{4\varepsilon}$ dependent on ε.

In turn, from the property (A.4) and the previous estimate we see that $Q_\varepsilon(\cdot, u_t^\rho)$ is bounded in $L^{\Psi_\varepsilon}((0,T)\times\mathbb{R}^n)$.

To obtain the estimate corresponding to (2.26) pass to the limit in (A.13) as $n\to\infty$. Hence from (A.12),

$$\sup_{t\in(0,T)} \|u_{tt}^\rho(t)\|_2^2 + \|\nabla u_t^\rho(t)\|_2^2 \le \text{Const.}(\|u_{0\rho}\|_{H^2(\mathbb{R}^n)} + \|u_{1\rho}\|_{H^1(\mathbb{R}^n)} + \||u_{1\rho}|^{m-2}|u_{1\rho}\|_2 + \|g_\rho(0)\|_2^2 + +\||(g_\rho)_t|\|_2^2). \qquad \text{(A.16)}$$

Thus, in turn

$$\begin{aligned} &u^\rho \to u \qquad \text{weakly}^* \text{ in } L^\infty(0,T;H^1(\mathbb{R}^n)), && \text{(A.17)}\\ &u_t^\rho \to u_t \qquad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\mathbb{R}^n)) \text{ and weakly in } L^{\Phi_\varepsilon}((0,T)\times\mathbb{R}^n), \\ &Q_\varepsilon(\cdot,\cdot,u_t^\rho) \to \chi \qquad \text{weakly in } L^{\Psi_\varepsilon}((0,T)\times\mathbb{R}^n), \end{aligned}$$

while also (2.28) is still valid.

Now, using (A.8) we can show that u is solution of

$$u_{tt} - \Delta u + \chi - g = 0 \qquad \text{in } H^1(\mathbb{R}^n) + L^{\Psi_\varepsilon}(\mathbb{R}^n)$$

in the distribution sense and hence almost everywhere. Then Theorem 3 implies that $u(0) = u_0$ and $u_t(0) = u_1$. Therefore, to complete the proof it only must be shown that $\chi = Q_\varepsilon(\cdot, u_t)$. But from the monotonicity of Q_ε and the arguments from [16],

$$\int_{(0,T)\times\mathbb{R}^n} (\chi - Q_\varepsilon(\cdot,w))(u_t - w) \ge 0 \qquad \text{for all } w \in L^{\Phi_\varepsilon}((0,T)\times\mathbb{R}^n).$$

We choose $w = u_t - \lambda w_1$, with $\lambda > 0$ and $w_1 \in L^{\Phi_\varepsilon}((0,T)\times\mathbb{R}^n)$. Hence,

$$\int_{(0,T)\times\mathbb{R}^n} (\chi - Q_\varepsilon(\cdot, u_t - \lambda w_1))w_1 \ge 0.$$

In turn, taking into account the continuity of the Nemitskii operator $u \mapsto Q_\varepsilon(\cdot,u)$, which follows by Lemma 4 and (A.15), and letting $\lambda\to 0$, we obtain

$$\int_{(0,T)\times\mathbb{R}^n} (\chi - Q_\varepsilon(\cdot, u_t))w_1 \ge 0.$$

Since this holds for arbitrary w_1, it follows that $\chi \equiv Q_\varepsilon(\cdot,u_t)$. $\square$

Appendix B. A density lemma

Lemma 5. *Let Φ be a Δ_2 regular N–function and suppose*

$$\phi \in \mathcal{K} := C([0,T]; L^2(\mathbb{R}^n)) \cap L^\Phi((0,T)\times\mathbb{R}^n) \cap H^1((0,T)\times\mathbb{R}^n).$$

Then there is a sequence $\phi_\varepsilon \in C_c^\infty(\mathbb{R}^{n+1})$ such that $\phi_\varepsilon|_{(0,T)\times\mathbb{R}^n} \to \phi|_{(0,T)\times\mathbb{R}^n}$ in $\mathcal{K}$.

Proof. Denote by $*$, $*_x$ and $*_t$ respectively space–time convolution, space convolution and time convolution. Now extend ϕ to $\mathbb{R}^{n+1}$ keeping the same regularity (for example by reflection on t) and construct an approximating sequence ϕ_ε in the form $\phi_\varepsilon := \eta_\varepsilon(\rho_\varepsilon * \phi)$. Here,

(a) $\eta_\varepsilon(x) = \eta(4\varepsilon^2|x|^2)$ for $x \in \mathbb{R}^n$, where η is a cut–off function such that $\eta \in C^\infty(\mathbb{R})$, $\eta(s) = 0$ if $|s| \geq 1$, $\eta(s) = 1$ if $|s| \leq 1/2$, $\|\eta\|_\infty = 1$;

(b) $\rho_\varepsilon(t,x) = \sigma_\varepsilon(x)\gamma_\varepsilon(t)$, where γ_ε and σ_ε are standard mollifiers in $\mathbb{R}$ and $\mathbb{R}^n$ respectively; of course then ρ_ε is a mollifier in $\mathbb{R}^{n+1}$.

It is well known that $\rho_\varepsilon * \phi|_{(0,T)\times\mathbb{R}^n} \to \phi|_{(0,T)\times\mathbb{R}^n}$ (see [4]) in $H^1((0,T)\times\mathbb{R}^n)$. From this, together with convolution arguments, it follows that $\phi_\varepsilon|_{(0,T)\times\mathbb{R}^n} \to \phi|_{(0,T)\times\mathbb{R}^n}$ in $H^1((0,T)\times\mathbb{R}^n)$. Next (see [1])

$$\rho_\varepsilon * \phi|_{(0,T)\times\mathbb{R}^n} \to \phi|_{(0,T)\times\mathbb{R}^n} \quad \text{in } L^\Phi((0,T)\times\mathbb{R}^n). \tag{B.1}$$

Denote by $|||\cdot|||_\Phi$ the norm in $L^\Phi((0,T)\times\mathbb{R}^n)$. Monotonicity of Φ and (3.15) then yield $|||\tilde\eta v|||_\Phi \leq \|\tilde\eta\|_\infty |||v|||_\Phi$, for all $v \in L^\Phi((0,T)\times\mathbb{R}^n)$ and $\tilde\eta \in L^\infty(\mathbb{R}^n)$. Hence,

$$\begin{aligned} |||\phi_\varepsilon - \phi|||_\Phi = |||\eta_\varepsilon(\rho_\varepsilon * \phi) - \phi|||_\Phi &\leq |||\eta_\varepsilon(\rho_\varepsilon * \phi) - \eta_\varepsilon\phi|||_\Phi + |||\eta_\varepsilon\phi - \phi|||_\Phi \\ &\leq |||\rho_\varepsilon * \phi - \phi|||_\Phi + |||\eta_\varepsilon\phi - \phi|||_\Phi. \end{aligned} \tag{B.2}$$

By the Δ_2–regularity of Φ we have $\Phi(|\eta_\varepsilon\phi - \phi|) \leq \text{Const.}\ \Phi(|\phi|)$. Moreover, $\eta_\varepsilon\phi \to \phi$ a.e. on $(0,T)\times\mathbb{R}^n$. Together with the continuity of Φ and the Lebesgue Dominated Convergence Theorem, this implies

$$\int_{(0,T)\times\mathbb{R}^n} \Phi(|\eta_\varepsilon\phi - \phi|) \to 0.$$

Recalling (A.5), this gives $\eta_\varepsilon\phi \to \phi$ in $L^\Phi((0,T)\times\mathbb{R}^n)$. Then from (B.1) and (B.2), we get $\phi_\varepsilon|_{(0,T)\times\mathbb{R}^n} \to \phi|_{(0,T)\times\mathbb{R}^n}$ in $L^\Phi((0,T)\times\mathbb{R}^n)$.

To prove the convergence of ϕ_ε in $C([0,T]; L^2(\mathbb{R}^n))$ we note that

$$\rho_\varepsilon * \phi = \sigma_\varepsilon *_x (\gamma_\varepsilon *_t \phi).$$

Moreover, from the arguments of [4], it follows that $\|\gamma_\varepsilon *_t \phi(t) - \phi(t)\|_2 \to 0$ uniformly in $[0,T]$, as $\varepsilon \to 0$. Hence, for all $t \in [0,T]$,

$$\begin{aligned}\|\rho_\varepsilon * \phi(t) - \phi(t)\|_2 &\le \|\sigma_\varepsilon *_x (\gamma_\varepsilon *_t \phi)(t) - \sigma_\varepsilon *_x \phi(t)\|_2 + \|\sigma_\varepsilon *_x \phi(t) - \phi(t)\|_2 \\ &\le \|\gamma_\varepsilon *_t \phi(t) - \phi(t)\|_2 + \|\sigma_\varepsilon *_x \phi(t) - \phi(t)\|_2 \to 0 \qquad \text{(B.3)}\end{aligned}$$

as $\varepsilon \to 0$. Now, for $s,t \in [0,T]$

$$\|\rho_\varepsilon * \phi(s) - \rho_\varepsilon * \phi(t)\|_2 \le \|\phi(s) - \phi(t)\|_2.$$

In turn, by (B.3) and the Arzelá–Ascoli Theorem one obtains (up to a subsequence) $\|\rho_\varepsilon * \phi(t) - \phi(t)\|_2 \to 0$ uniformly for $t \in [0,T]$. Moreover, $\|\eta_\varepsilon \phi(t) - \phi(t)\|_2 \to 0$ pointwise in $[0,T]$ as $\varepsilon \to 0$. Next, for $s,t \in [0,T]$,

$$\|\eta_\varepsilon \phi(s) - \eta_\varepsilon \phi(t)\|_2 \le \|\phi(s) - \phi(t)\|_2.$$

Consequently, applying the Arzelá–Ascoli Theorem again, it follows that (up to a subsequence), $\|\eta_\varepsilon \phi(t) - \phi(t)\|_2 \to 0$ uniformly in $[0,T]$ as $\varepsilon \to 0$. Since

$$\begin{aligned}\|\phi_\varepsilon(t) - \phi(t)\|_2 &\le \|\eta_\varepsilon(\rho_\varepsilon * \phi)(t) - \eta_\varepsilon \phi(t)\|_2 + \|\eta_\varepsilon \phi(t) - \phi(t)\|_2 \\ &\le \|\rho_\varepsilon * \phi(t) - \phi(t)\|_2 + \|\eta_\varepsilon \phi(t) - \phi(t)\|_2\end{aligned}$$

for all $t \in [0,T]$, we derive $\phi_\varepsilon \to \phi$ in $C([0,T]; L^2(\mathbb{R}^n))$. This completes the proof. $\square$

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