



Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms

Grozdena Todorova^{*,1}

Institute of Mathematics, Bulgarian Academy of Sciences, Boul. Acad. G. Bonchev bl.8, Sofia 1113, Bulgaria

Received 12 January 1998; accepted 10 August 1998

Keywords: Nonlinear wave equation; Global existence; Blow up

1. Introduction

In this paper, we study the Cauchy problem for the nonlinear wave equation with nonlinear damping and source terms

$$\square u + au_t|u_t|^{m-1} = bu|u|^{p-1} \quad \text{for } (t, x) \in \mathbf{R} \times \mathbf{R}^n, \quad (1.1)$$

with initial data

$$u(0, x) = f(x), \quad u_t(0, x) = g(x). \quad (1.2)$$

Here $a, b > 0$ and $p > 1, m > 1$.

In case of IBVP, in a bounded domain $\Omega \subset \mathbf{R}^n$ with Dirichlet boundary conditions, the following results are known:

1. When $a = 0$, it is proved (see [1, 3, 8, 14, 16]) that the solution blows up in finite time for sufficiently large initial data.
2. When $b = 0$, Haraux and Zuazua [5] and Kopackova [7] prove the global existence result for large initial data.

The behavior of the solution of Eq. (1.1) with nonlinear source and linear damping (case $m = 1$) in an abstract setting was considered by Levine in [9]. More precisely, he showed that the solutions with negative initial energy are nonglobal.

* Tel.: 00-359-2-979-3851; fax: 003592-9713649.

E-mail address: grozdena@bas.bg (G. Todorova)

¹ Work partially supported by Bulgarian Science Foundation Grant MM-516.

For bounded domains the interaction between nonlinear damping and source terms for Eq. (1.1) was studied by Georgiev and Todorova [4]. They prove that if $m \geq p$, a global weak solution exists for any initial data, while if $1 < m < p$ the solution blows up in finite time when the initial energy $E(0) = 1/2 \|\nabla f\|_2^2 + 1/2 \|g\|_2^2 - 1/(p+1) \|f\|_{p+1}^{p+1}$ is sufficiently negative.

Levine and Serrin in [10] consider the abstract version of Eq. (1.1), namely $[P(u_t)]_t + A(u) + Q(t, u_t) = F(u)$, on appropriate Banach spaces, where Q is as a damping term and F is as a driving force. The application of their arguments to the Eq. (1.1) shows that for any negative initial energy $E(0)$ the solution blows up in finite time.

Ikehata [6] and Ohta [12] consider the solutions of the Eqs. (1.1)–(1.2) with small positive initial energy, using the so-called “potential well” theory introduced by Payne and Sattinger in [14]. Using the same functional as in [4], Ono in [13] shows the blow up result for the nonlinear Kirchhoff equation with nonlinear damping and source terms. His result is also a consequence of [10].

In all above treatments the underlying domain is assumed to be bounded. The boundedness of the domain Ω is essential because of the usage of the boundedness of the injection $L^p(\Omega) \subset L^q(\Omega)$ when $1 \leq q \leq p$. In this paper, we consider the interaction between the nonlinear damping and source terms for the Cauchy problem (1.1)–(1.2) when $a = b = 1$. Let us mention that the special polynomial form of the source and the damping is not essential. The results could be extended to the case of more general nonlinearities under suitable assumptions. The lack of the injection $L^p(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ shall be compensated by the usage of a special weighted functional.

We shall restrict ourselves in the case of compactly supported data. This restriction is crucial for the blow-up part of the paper. It is not essential for the global existence part, where it is only due to the usage of a local existence result for compactly supported data. The local existence theorem could be proved removing the requirement for the compact support of the data.

We shall prove the following theorems:

Theorem 1. *If $p \leq m$ and*

$$1 < p \leq \frac{n}{n-2}, \quad \text{for } n \geq 3, \quad (1.3)$$

$$1 < p, \quad \text{for } n \leq 2. \quad (1.4)$$

Then, for any compactly supported data

$$f \in H^1(\mathbb{R}^n), \quad g \in L^2(\mathbb{R}^n), \quad (1.5)$$

the Cauchy problem (1.1)–(1.2) has a unique global solution, such that

$$u(t, x) \in C([0, T]; H^1(\mathbb{R}^n)),$$

$$u_t(t, x) \in C([0, T]; L^2(\mathbb{R}^n)) \cap L^{m+1}([0, T] \times \mathbb{R}^n)$$

for any $T > 0$.

Theorem 2. *Let $1 < m < p$, and let the conditions (1.3)–(1.5) of Theorem 1 be fulfilled. Then the weak solution of (1.1)–(1.2) blows up in finite time if the initial energy $E(0)$ is negative and the additional restriction on m , namely $m > np/(n + p + 1)$, is fulfilled. When $1 < m \leq np/(n + p + 1)$, the solution blows up for sufficiently negative initial energy $E(0)$ and $\int fg \, dx \geq 0$.*

Let us mention that the requirement $\int fg \, dx \geq 0$ is one of the possible requirements, which together with the essential condition for large negative initial energy $E(0)$ assures the blow up of the solution in the second part of the above blow up Theorem.

In particular, the following Corollary is fulfilled.

Corollary 1.1. *Let $1 < m \leq np/(n + p + 1)$. For any compactly supported data $f \in H^1(R^n)$, $g \in L^2(R^n)$ with initial energy $E(0) < -1$, after some rescaling with convenient constant $S > 1$, the weak solution of (1.1)–(1.2) (for the data Sf, Sg) blows up in finite time.*

A natural question arises: could some new effects due to the unboundedness of the domain exist for small $E(0) < 0$ under the “wall” defined by the following inequality $m \leq np/(n + p + 1)$? The new effects mean possible global existence results for small $E(0) < 0$ and $m \leq np/(n + p + 1)$. The blow up result of Levine [9] shows that in case of linear damping new effects coming from the unboundedness of the domain do not exist. In the case of nonlinear damping, we use the following way to overcome the lack of the estimate of the norm $\|u\|_2$. Assume that the global existence results exist for some small $E(0) < 0$ and $m \leq np/(n + p + 1)$. We consider the Eq. (1.1) with an additional decaying linear damping, small $E(0) < 0$ and $m \leq np/(n + p + 1)$. Then, in the above perturbed equation with decaying linear damping, the global existence effects should be manifested even stronger. We shall prove that the solution of the perturbed equation blows up for any negative initial energy $E(0)$. This means that for the Eq. (1.1) there are no new effects coming from the unboundedness of the domain, i.e. the “wall” in Theorem 2 is artificially created.

So, we shall consider the Cauchy problem for the wave equation with nonlinear damping and source terms with decaying linear perturbation of the type $q^2(x)u$

$$\square u + u_t |u_t|^{m-1} + q^2(x)u = u|u|^{p-1} \quad \text{for } (t, x) \in \mathbf{R} \times \mathbf{R}^n, \quad (1.6)$$

where $\lim_{|x| \rightarrow \infty} q(x) = 0$. More precisely, $q(x)$ is a locally bounded measurable function on R^n , which satisfies

$$q(x) \geq s(1 + |x|)^{-\eta}, \quad (1.7)$$

where $\eta \geq 0$ and $s > 0$. The decay rate η of $q(x)$ satisfies the assumption

$$\eta \leq \min \left(\frac{1}{p+1}, \frac{(p-1)m}{2(p-m)} \right). \quad (1.8)$$

For the Cauchy problem (1.6) with decaying linear perturbation we shall prove the following blow up theorem.

Theorem 3. *Let $1 < m < p$, the conditions (1.3)–(1.5) of Theorem 1 and the Assumptions (1.7)–(1.8) be fulfilled. Then the weak solution of Eq. (1.6) blows up in finite time if the initial energy $E(0)$ is negative.*

Very recently the author learned that Levine et al. [11] consider the following IVP:

$$\square u + Q(t, x, u_t) = f(u) \quad \text{for } (t, x) \in \mathbf{R} \times \mathbf{R}^n, \quad (1.9)$$

with compactly supported data under suitable assumptions for the terms $Q(t, x, u_t)$ and $f(u)$. They establish the blow up result for weak solutions with negative initial energy $E(0)$ when the parameters of the problem (1.9) satisfy additional restrictions. Without these restrictions, the authors prove the blow up of solutions with sufficiently negative initial energy $E(0)$. The question about the nature of these additional restrictions on the parameters of the problem (1.9), i.e. whether they are effects of the unboundedness of the domain or they are created by the methods, is not treated in their paper. As remarked above, some of the results in this paper and their results are overlapping in some respects; the work, however, in both cases is entirely independent.

2. Global existence result for large data

For problem (1.1)–(1.2) we have the following local existence theorem:

Theorem 4. *Let $m > 1$ and*

$$1 < p \leq \frac{n}{(n-2)}, \quad \text{for } n \geq 3; \quad 1 < p, \quad \text{for } n \leq 2.$$

Then, for any data $f \in H^1(\mathbf{R}^n)$, $g \in L^2(\mathbf{R}^n)$ with compact support, the Cauchy problem (1.1)–(1.2) has a unique solution, such that

$$u(t, x) \in C([0, T]; H^1(\mathbf{R}^n)),$$

$$u_t(t, x) \in C([0, T]; L^2(\mathbf{R}^n)) \cap L^{m+1}([0, T] \times \mathbf{R}^n)$$

for T small enough.

The proof follows directly from the argument of [4].

Proof of the global existence Theorem 1. Once the local existence theorem has been obtained, the continuation principle can be proved as in [15].

Denote

$$\Phi(t) = \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{p+1} \|u\|_{p+1}^{p+1}.$$

Then

$$\dot{\Phi}(t) = -\|u_t\|_{m+1}^{m+1} + 2 \int uu_t |u|^{p-1} dx. \quad (2.1)$$

To estimate the integral in Eq. (2.1) we use the Holder and Young inequalities and get

$$\left| \int uu_t |u|^{p-1} dx \right| \leq \|u\|_{p+1}^p \|u_t\|_{p+1} \leq C(\varepsilon) \|u\|_{p+1}^{p+1} + \varepsilon \|u_t\|_{p+1}^{p+1}. \quad (2.2)$$

Now, from the convexity of the function (u^y/y) in y for $u \geq 0$ and for $y > 0$, we obtain

$$\frac{\|u_t\|_{p+1}^{p+1}}{p+1} \leq C_1 \frac{\|u_t\|_2^2}{2} + C_2 \frac{\|u_t\|_{m+1}^{m+1}}{m+1},$$

since $2 \leq p+1 \leq m+1$. In the above C_1 and C_2 are positive constants. Then, from Eqs. (2.1) and (2.2), we get

$$\dot{\Phi}(t) \leq -\|u_t\|_{m+1}^{m+1} + 2C_2\varepsilon \frac{\|u_t\|_{m+1}^{m+1}}{m+1} + 2C_1\varepsilon \frac{\|u_t\|_2^2}{2} + 2C(\varepsilon)\|u\|_{p+1}^{p+1}. \quad (2.3)$$

Choosing ε small enough in Eq. (2.3), we arrive at $\dot{\Phi}(t) \leq C\Phi(t)$.

Then the Gronwall lemma and the continuation principle complete the proof of the global existence result. \square

3. Blow up results

Proof of the blow up Theorem 2. For the case of bounded domain Ω , the functional

$$W(t) = H(t)^{1-\alpha} + \varepsilon \dot{F}(t), \quad (3.1)$$

where

$$H(t) = -\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$

$$F(t) = \|u\|_2^2$$

and the constants $\varepsilon > 0$ and $0 < \alpha < 1$ are conveniently chosen, was used in [4] to manifest the blow up property of the solution.

For the Cauchy problem we shall use the weighed functional

$$W(t) = (L+t)^A H(t)^{1-\alpha} + \varepsilon \dot{F}(t), \quad (3.2)$$

where the constants $\alpha < 1$, $A > 0$ and $\varepsilon > 0$ shall be chosen in the sequel. The constant L is such that $\text{supp}\{f(x), g(x)\} \subset \{|x| \leq L\}$, $L > 0$. The weight $(L+t)^A$ is used to compensate the lack of the injection $L^p(R^n) \subset L^q(R^n)$ when $1 \leq q \leq p$.

The function $H(t)$ is increasing since from the energy identity we have $\dot{H}(t) = \|u_t\|_{m+1}^{m+1}$. Let $H(0) > 0$. Then

$$0 < H(0) \leq H(t) \leq \|u\|_{p+1}^{p+1}. \quad (3.3)$$

We have

$$\dot{W}(t) = (L+t)^A(1-\alpha)H(t)^{-\alpha}\dot{H}(t) + A(L+t)^{A-1}H(t)^{1-\alpha} + \varepsilon\ddot{F}. \quad (3.4)$$

Thus, using the energy identity, we get

$$\begin{aligned} \dot{W}(t) &\geq (L+t)^A(1-\alpha)H(t)^{-\alpha}\dot{H}(t) \\ &+ \varepsilon \left[4\|u_t\|_2^2 + 4H(t) + 2\frac{p-1}{p+1}\|u\|_{p+1}^{p+1} - 2\int uu_t|u_t|^{m-1} \right]. \end{aligned} \quad (3.5)$$

To estimate the integral $\int uu_t|u_t|^{m-1}$ we use the Holder inequality, the compact support of the data and the finite speed of propagation for Eq. (1.1). So, we get

$$\int uu_t|u_t|^{m-1} \leq \|u\|_{m+1}\|u_t\|_{m+1}^m \leq \|u\|_{p+1} \left(\int_{B(L+t)} 1 \right)^{1/s} \|u_t\|_{m+1}^m, \quad (3.6)$$

where

$$1/(m+1) = 1/(p+1) + 1/s \quad (3.7)$$

and $B(L+t)$ is a ball with radius $L+t$ centered at the origin. Then we have

$$\begin{aligned} \int uu_t|u_t|^{m-1} &\leq C\|u\|_{p+1}^{1-(p+1)/(m+1)}\|u\|_{p+1}^{(p+1)/(m+1)}(L+t)^{n/s}\|u_t\|_{m+1}^m \\ &\leq H(t)^{(m-p)/[(p+1)(m+1)]}(C(\varepsilon_1)(L+t)^{[n(m+1)]/[sm]}\dot{H}(t) + \varepsilon_1\|u\|_{p+1}^{p+1}). \end{aligned} \quad (3.8)$$

In the above estimate we use the Young inequality and the assumption $m < p$. We indicate by C positive constants, which are possibly different from line to line. We choose the positive constant $\alpha < 1/2$ so that $\alpha < (p-m)/[(p+1)(m+1)]$. Let us mention that α shall be diminished additionally in the sequel. Denote $\beta = ((m-p)/[(p+1)(m+1)] + \alpha) < 0$. Then the properties (3.3) of the function $H(t)$ show that

$$H(t)^{(m-p)/[(p+1)(m+1)]} = H(t)^{-\alpha}H(t)^\beta \leq H(t)^{-\alpha}H(0)^\beta.$$

Thus, from Eq. (3.5), it follows that

$$\begin{aligned} \dot{W}(t) &\geq ((L+t)^A(1-\alpha) - \varepsilon C(\varepsilon_1)(L+t)^{n(m+1)/[sm]}H(0)^\beta)H(t)^{-\alpha}\dot{H}(t) \\ &+ \varepsilon \left[4\|u_t\|_2^2 + 4H(t) + \left(2\frac{p-1}{p+1} - \varepsilon_1H(0)^{(m-p)/[(p+1)(m+1)]} \right) \|u\|_{p+1}^{p+1} \right]. \end{aligned} \quad (3.9)$$

We choose

$$A = \frac{n(m+1)}{sm} = \frac{n(p-m)}{(p+1)m} \quad (3.10)$$

using Eq. (3.7). The constant ε_1 is fixed so that

$$\varepsilon_1 H(0)^{(m-p)/[(p+1)(m+1)]} < \frac{p-1}{p+1}. \quad (3.11)$$

Respectively, we choose ε so small that the inequalities

$$\varepsilon C(\varepsilon_1) H(0)^\beta < \frac{1}{4} \quad (3.12)$$

and

$$W(0) = L^A H(0)^{1-\alpha} + \varepsilon \dot{F}(0) > 0 \quad (3.13)$$

are fulfilled.

Finally, we get

$$\dot{W}(t) \geq C[\|u_t\|_2^2 + H(t) + \|u\|_{p+1}^{p+1}]. \quad (3.14)$$

Now, we shall diminish α in an appropriate way so that the following strong inequality:

$$\dot{W}(t) \geq C W(t)^{1/(1-\alpha)} (L+t)^{(-A_1)/(1-\alpha)} \quad (3.15)$$

is fulfilled for suitably chosen in the sequel constant A_1 , $A_1 \geq A$. Once estimate (3.15) has been proved, we obtain in a standard way the finite time blow-up for u . Taking into account Eq. (3.14), it is sufficient to show that

$$\|u_t\|_2^2 + H(t) + \|u\|_{p+1}^{p+1} \geq C W(t)^{1/(1-\alpha)} (L+t)^{(-A_1)/(1-\alpha)}. \quad (3.16)$$

Really, we have $H(t)^{(1-\alpha)/(1-\alpha)} \leq \|u\|_{p+1}^{p+1}$. Now, we shall verify the estimate

$$(L+t)^{(-A_1)/(1-\alpha)} \left(\int u u_t \right)^{1/(1-\alpha)} \leq C \|u\|_{p+1}^{p+1}. \quad (3.17)$$

Denote $1/(1-\alpha) = 1 + \delta$, where $\delta > 0$. Then, using the Holder and the Young inequalities, we have

$$\left(\int u u_t \right)^{1+\delta} \leq C (\|u\|_{p+1}^{(1+\delta)\lambda} + \|u_t\|_2^{(1+\delta)\mu}) \left(\int_{B(L+t)} 1 \right)^{(1+\delta)/r}, \quad (3.18)$$

where $1/\mu + 1/\lambda = 1$ and

$$1/r = 1/2 - 1/(p+1) = \frac{p-1}{2(p+1)}. \quad (3.19)$$

We choose μ such that $(1+\delta)\mu=2$. Then $\mu=2/(1+\delta)<2$ and $\lambda>2$. Let us mention that if δ is small, μ and λ are near 2. Therefore, it is possible to choose δ so small that

$$2<(1+\delta)\lambda<p+1. \quad (3.20)$$

Then,

$$\|u\|_{p+1}^{(1+\delta)\lambda} = \|u\|_{p+1}^{p+1} \|u\|_{p+1}^{-K} \leq \|u\|_{p+1}^{p+1} H(0)^{-K/(p+1)}, \quad (3.21)$$

where $K=p+1-(1+\delta)\lambda$ is a positive constant. Now, from Eq. (3.18), we have

$$\left(\int uu_t\right)^{1+\delta} \leq (\|u\|_{p+1}^{p+1} H(0)^{-K/(p+1)} + \|u_t\|_2^2) \left(\int_{B(L+t)} 1\right)^{(1+\delta)/r}, \quad (3.22)$$

which completes the proof of Eq. (3.15) with

$$A_1 = \max(A, n/r) = \max\left(\frac{n(p-m)}{(p+1)m}, n/r\right). \quad (3.23)$$

We consider two cases:

Case 1: The integral $\int_0^\infty (L+t)^{(-A_1)(1+\delta)} dt = \infty$. This happens when $(-A_1)(1+\delta)+1>0$. Since $n/r = [n(p-1)]/[2(p+1)]$ (from Eq. (3.19)), the inequality $1>n/r$ is obvious when $n\leq 2$. When $n>2$ the inequality $1>n/r$ is equivalent to $p<(n+2)/(n-2)$ and it trivially follows from the Sobolev restriction $p\leq n/(n-2)$. So we choose δ so small that the inequality $1>n/r(1+\delta)$ is still valid. The inequality $1>A$ (see Eq. (3.10)) follows from the assumption $m>(np)/(n+p+1)$. We diminish δ so that the inequality $1>A(1+\delta)$ is still fulfilled. The final choice for δ is δ so small that the inequalities $2<(1+\delta)\lambda<p+1$, $1>n/r(1+\delta)$ and $-[n(p-m)]/[(p+1)m](1+\delta)+1>0$ are fulfilled. This completes the proof of the blow up result in this case.

Case 2: The integral $J = \int_0^\infty (L+t)^{(-A_1)(1+\delta)} dt < \infty$. This takes place when $-A_1(1+\delta)+1<0$. Since δ could be chosen as small as necessary (see Eq. (3.20)), we get $-A_1+1\leq 0$. From Eq. (3.23) this leads to the restriction $m\leq np/(n+p+1)$. Since $H(0)>1$, the constants ε_1 in Eq. (3.11) and ε in Eq. (3.12) can be chosen to be independent of $H(0)$. Inequality (3.13) is automatically fulfilled, provided $\dot{F}(0) = \int_{B(L)} fg dx \geq 0$. Finally, in Eq. (3.15) the constant C can be chosen to be also independent of $H(0)$, (see Eq. (3.22), where $H(0)^{-K/(p+1)}$ can be replaced by 1). Suppose that $u(t, x)$ is a solution of Eqs. (1.1)–(1.2) for all $t\geq 0$. Then, integrating Eqs. (3.15) from $(0, \infty)$, we get $W(0) \leq \{1/(\delta CJ)\}^{1/\delta}$ (note that from Eq. (3.13) $W(0)>0$). Since $W(0) = L^A H(0)^{1-\alpha} + \varepsilon \dot{F}(0)$, in the case $\dot{F}(0)\geq 0$ the above inequality leads to $H(0) \leq (C_1/L^A)^{1+\delta}$, where $C_1 = \{1/(\delta CJ)\}^{1/\delta}$. So, if $H(0)$ is sufficiently large, namely $H(0) > \max\{1, \{C_1/L^A\}^{1+\delta}\}$, the solution blows up in finite time. This completes the proof to the second part of the blow up Theorem 2. \square

Proof of the Corollary 1.1. After rescaling the initial data $Sf(x), Sg(x)$ with constant $S>1$, we set $W_S(0) = L^A H_S^{1-\alpha}(0) + \varepsilon S^2 \dot{F}(0)$, where $H_S(0) = -1/2S^2 \|g\|_2^2 - 1/2S^2 \|\nabla f\|_2^2 + 1/(p+1)S^{p+1} \|f\|_{p+1}^{p+1}$. So $H_S(0) > S^{p+1} H(0)$. Then the increasing of $H(0)$ up to

$H_S(0)$ in Eq. (3.9) is possible due to Eqs. (3.11) and (3.12). The choice of δ in Eq. (3.20) leads to $(1+\delta)2 < p+1$, i.e. $(p+1)(1-\alpha) > 2$. Then the inequality (3.13) written for $W_S(0)$ looks like $W_S(0) = L^A H_S(0)^{1-\alpha} + \varepsilon S^2 \dot{F}(0) > S^2 W(0) > 0$. Finally, we have

$$\left\{ \frac{1}{\delta C J} \right\}^{1/\delta} > W_S(0) \geq L^A S^{(p+1)(1-\alpha)} H^{1-\alpha}(0) + \varepsilon S^2 \dot{F}(0). \quad (3.24)$$

The above inequality is violated for sufficiently large S . \square

Proof of the blow up Theorem 3. For the Cauchy problem (1.6) we use the following weighted functional:

$$W(t) = \int_0^t (L + \tau)^A H(\tau)^{1-\alpha} d\tau + \varepsilon (F(t))^{1-B}, \quad (3.25)$$

where $H(t) = -1/2 \|u_t\|_2^2 - 1/2 \|\nabla u\|_2^2 - 1/2 \|qu\|_2^2 + 1/(p+1) \|u\|_{p+1}^{p+1}$, $F(t) = \|u\|_2^2$ and the constants $\alpha < 1$, $A > 0$, $0 < B < 1$ and $\varepsilon > 0$ shall be chosen in the sequel. The constant L is such that $\text{supp}\{f(x), g(x)\} \subset \{|x| \leq L\}$, $L > 0$.

Due to the finite propagation speed we have

$$\|u\|_2 \leq C(L+t)^\eta \|qu\|_2. \quad (3.26)$$

The function $H(t)$ is increasing since $\dot{H}(t) = \|u_t\|_{m+1}^{m+1}$. Let $H(0) > 0$. Then

$$0 < H(0) \leq H(t) \leq \|u\|_{p+1}^{p+1}. \quad (3.27)$$

We have

$$\dot{W}(t) = (L+t)^A H(t)^{1-\alpha} + \varepsilon(1-B) F^{-B} \int uu_t dx. \quad (3.28)$$

So, we arrive at

$$\begin{aligned} \ddot{W}(t) &\geq (1-\alpha)(L+t)^A H(t)^{-\alpha} \dot{H}(t) + \frac{(1-B)\varepsilon}{F(t)^B} \\ &\quad \times \left[4\|u_t\|_2^2 + 4H(t) + 2\frac{p-1}{p+1} \|u\|_{p+1}^{p+1} - 2 \int uu_t |u_t|^{m-1} \right] \\ &\quad - \frac{B(1-B)\varepsilon}{F^{1+B}} \left(\int uu_t \right)^2. \end{aligned} \quad (3.29)$$

To estimate the integral $\int uu_t |u_t|^{m-1}$, we use the Holder inequality and the so called interpolation inequality

$$\|f\|_r \leq \|f\|_p^\delta \|f\|_q^{1-\delta},$$

where

$$\frac{1}{r} = \frac{\delta}{p} + \frac{1-\delta}{q}$$

for $p \leq r \leq q$ and $0 \leq \delta \leq 1$ (see [2]). Since $2 < m + 1 < p + 1$, we get

$$\left| \int uu_t |u_t|^{m-1} \right| \leq \|u\|_{m+1} \|u_t\|_{m+1}^m \leq \|u\|_2^\delta \|u\|_{p+1}^{1-\delta} \|u_t\|_{m+1}^m, \quad (3.30)$$

where

$$\delta = \left(\frac{1}{m+1} - \frac{1}{p+1} \right) \bigg/ \left(\frac{1}{2} - \frac{1}{p+1} \right). \quad (3.31)$$

Then we have

$$\begin{aligned} \|u\|_2^\delta \|u\|_{p+1}^{1-\delta} \|u_t\|_{m+1}^m &\leq \|u_t\|_{m+1}^m \|u\|_2^{\delta_1} \|u\|_{p+1}^{1-\delta} \|u\|_2^{\delta-\delta_1} \\ &\leq C \|u_t\|_{m+1}^m \|u\|_{p+1}^{1-(p+1)/(m+1)-\delta+((p+1)/2)\delta_1} (L+t)^{\delta_1 \eta} \\ &\quad \times \|u\|_2^{\delta-\delta_1} \|u\|_{p+1}^{(p+1)/(m+1)}. \end{aligned} \quad (3.32)$$

In the above estimate the positive constant $\delta_1 < \delta$ shall be chosen in the sequel.

Denote $\lambda = 1 - (p+1)/(m+1) - \delta + \delta_1(p+1)/2$. Then, using Eq. (3.31), we get

$$\lambda = 1 - \frac{p+1}{m+1} - \delta + \delta \frac{p+1}{2} + \frac{p+1}{2} (\delta_1 - \delta) = 0 + \frac{p+1}{2} (\delta_1 - \delta) < 0. \quad (3.33)$$

By means of Eq. (3.27) and the Young inequality we estimate the right-hand side of Eq. (3.32) and obtain

$$\begin{aligned} \|u\|_2^\delta \|u\|_{p+1}^{1-\delta} \|u_t\|_{m+1}^m &\leq \varepsilon_1 \|u\|_{p+1}^{p+1} + C(\varepsilon_1) H(t)^{\lambda(m+1)/[(p+1)m]} \\ &\quad \times \|u_t\|_{m+1}^{m+1} (L+t)^{\delta_1 \eta(m+1)/m} \|u\|_2^{(\delta-\delta_1)(m+1)/m}. \end{aligned} \quad (3.34)$$

The choice for $B < \frac{1}{2}$ is

$$2B = (\delta - \delta_1)(m+1)/m. \quad (3.35)$$

The above choice for B leads to the requirement $(\delta - \delta_1)(m+1)/m < 1$, i.e. $\delta_1 > \delta - m/(m+1)$. So,

$$\delta > \delta_1 > \max(0, \delta - m/(m+1)). \quad (3.36)$$

We choose the positive constant α

$$\alpha = \frac{-\lambda(m+1)}{(p+1)m}. \quad (3.37)$$

So, $\alpha = B$ (remind Eqs. (3.33) and (3.35)) The requirement $\alpha < 1$ is equivalent to $m > 1$.

Thus, from Eq. (3.29) together with Eq. (3.34), we get

$$\begin{aligned}\ddot{W}(t) \geq & ((L+t)^A(1-\alpha) - \varepsilon C(\varepsilon_1)(L+t)^{[\delta_1(m+1)\eta]/m})H(t)^{-\alpha}\dot{H}(t) \\ & + \frac{(1-B)\varepsilon}{F(t)^B} \left[4\|u_t\|_2^2 + 4H(t) + \left(2\frac{p-1}{p+1} - \varepsilon_1 \right) \|u\|_{p+1}^{p+1} \right] \\ & - \frac{B(1-B)\varepsilon}{F^{1+B}} \left(\int uu_t \right)^2.\end{aligned}\quad (3.38)$$

For the integral $I = 1/(F^{1+B})(\int uu_t)^2$ we have

$$I \leq \frac{\|u\|_2^2 \|u_t\|_2^2}{\|u\|_2^2 \|u\|_2^{2B}} = \frac{\|u_t\|_2^2}{\|u\|_2^{2B}}, \quad (3.39)$$

i.e.

$$IB \leq \frac{\|u_t\|_2^2}{\|u\|_2^{2B}},$$

since $B < 1$.

We fix

$$A = \frac{(m+1)\delta_1\eta}{m}. \quad (3.40)$$

The final choice for δ_1 is such that $A = [(m+1)\delta_1\eta]/m < 1$. This leads to $\delta_1 < \delta$ by means of Assumption (1.8) for the decay rate η together with Eq. (3.31). So, choice (3.36) for δ_1 is not affected.

The constant ε_1 is chosen so that

$$2\frac{p-1}{p+1} - \varepsilon_1 > 0.$$

Respectively, we fix ε so that the inequalities

$$(1-\alpha) - \varepsilon C(\varepsilon_1) \geq 0 \quad (3.41)$$

and

$$\dot{W}(0) = L^A H(0)^{1-\alpha} + \varepsilon(1-B)F(0)^{-B}\dot{F}(0) > 0 \quad (3.42)$$

are fulfilled.

Finally, we get

$$\ddot{W}(t) \geq C[\|u_t\|_2^2 + H(t) + \|u\|_{p+1}^{p+1}/F^B]. \quad (3.43)$$

The inequality (3.43) together with Eq. (3.42) show that $\dot{W}(t) > 0$ for $t \geq 0$.

Using Eq. (3.43), we shall prove the following rough estimate:

$$\ddot{W}(t) \geq C(L+t)^{-A_1} \dot{W}(t), \quad (3.44)$$

where the constant $A_1 \geq A$ shall be fixed later. Since

$$\dot{W}(t) = (L + t)^A H(t)^{1-\alpha} + \varepsilon(1 - B)F^{-B} \int uu_t \, dx, \quad (3.45)$$

we start with the estimate of the term $(L + t)^A H(t)^{1-\alpha}$. So, we have

$$\begin{aligned} (L + t)^A H(t)^{1-\alpha} F^B &\leq C \|u\|_{p+1}^{(p+1)(1-\alpha)+B(p+1)} (L + t)^{\eta 2B+A} \\ &\leq C \|u\|_{p+1}^{p+1} (L + t)^{\eta 2B+A}, \end{aligned} \quad (3.46)$$

since $\alpha = B$. For the integral $\int uu_t$ we get

$$\int uu_t \leq \|u\|_2 \|u_t\|_2 \leq C \|u\|_{p+1}^{p+1} (L + t)^\eta. \quad (3.47)$$

This completes the proof of Eq. (3.44) with $A_1 = \max(A + 2\eta B, \eta)$. Integrating the rough estimate (3.44), we get

$$\exp(C(L + t)^{-A_1+1}) \leq \dot{W}(t).$$

From this inequality the function $\dot{W}(t)$ has an exponential growth if $-A_1 + 1 > 0$. Due to the Assumption (1.8) $\eta < 1$. From Eqs. (3.35) and (3.40) we get

$$\begin{aligned} \eta 2B + A &= \eta(\delta - \delta_1)(m + 1)/m + \delta_1 \eta(m + 1)/m \\ &= \eta \delta(m + 1)/m = \frac{\eta(p - m)2}{(p - 1)m} < 1. \end{aligned} \quad (3.48)$$

The last inequality is due to the Assumption (1.8) for the decay rate η .

The exponential growth of $\dot{W}(t)$ leads to the exponential growth of the norm $\|u\|_{p+1}^{p+1}$. Really,

$$H(t)^{1-\alpha} \leq H(t)H(0)^{-\alpha} \leq C \|u\|_{p+1}^{p+1}. \quad (3.49)$$

Then recalling Eq. (3.35), we have

$$F^{-B} \int uu_t \leq \|u\|_2^{1-2B} \|u_t\|_2 \leq C \|u\|_{p+1}^{p+1} (L + t)^{\eta(1-2B)},$$

which together with Eq. (3.49) give

$$\dot{W}(t) \leq C \|u\|_{p+1}^{p+1} (L + t)^{A_2}$$

with $A_2 = \max(A, \eta(1 - 2B))$. The exponential growth of the norm $\|u\|_{p+1}^{p+1}$ gives us the possibility to compensate in a more delicate way the polynomial tails coming from the Holder inequality in the unbounded domain. We do not have to use the weighted functionals; instead of this, we work with functionals without weights and compensate the polynomial tails using the exponential growth of the norm $\|u\|_{p+1}^{p+1}$. The advantage of the second way is that it creates no “wall” and the whole region of negative initial energies $E(0)$ could be covered.

Finally, to manifest the blow up property of the solution for all negative initial energies $E(0)$, we use the functional without weights, namely

$$W(t) = H(t)^{1-\theta} + \zeta \dot{F}(t), \quad (3.50)$$

where $H(t) = -1/2\|u_t\|_2^2 - 1/2\|\nabla u\|_2^2 - 1/2\|qu\|_2^2 + 1/(p+1)\|u\|_{p+1}^{p+1}$, $F(t) = \|u\|_2^2$ and the constants $\zeta > 0$ and $0 < \theta < 1$ shall be fixed appropriately.

Then, following the arguments from Theorem 2, we can write

$$\begin{aligned} \dot{W}(t) &\geq (1-\theta)H(t)^{-\theta}\dot{H}(t) \\ &+ \zeta \left[4\|u_t\|_2^2 + 4H(t) + 2\frac{p-1}{p+1}\|u\|_{p+1}^{p+1} - 2\int uu_t|u_t|^{m-1} \right]. \end{aligned} \quad (3.51)$$

With the estimate of the integral $\int uu_t|u_t|^{m-1}$ we shall demonstrate how the polynomial tales could be compensated in a better way using the exponential growth of the norm $\|u\|_{p+1}^{p+1}$.

So, we have

$$\begin{aligned} \int uu_t|u_t|^{m-1} &\leq C\|u\|_{p+1}^{1-(p+1)/(m+1)}\|u\|_{p+1}^{(p+1)/(m+1)}(L+t)^{n/s}\|u_t\|_{m+1}^m \\ &\leq H(t)^{-\theta}C(\varepsilon_1)\frac{(L+t)^{[n(m+1)]/[sm]}}{\exp(C(L+t)^{-A_1+1}\xi)}\dot{H}(t) \\ &+ \varepsilon_1 H(0)^{(m-p)/[(p+1)(m+1)]}\|u\|_{p+1}^{p+1}, \end{aligned} \quad (3.52)$$

where

$$1/(m+1) = 1/(p+1) + 1/s \quad (3.53)$$

and $1-(p+1)/(m+1) = -(p+1)(\theta+\xi)$ with $0 < \xi$ and $0 < \theta < 1/(m+1) - 1/(p+1)$. This leads to

$$\int uu_t|u_t|^{m-1} \leq CH(t)^{-\theta}C(\varepsilon_1)\dot{H}(t) + \varepsilon_1 H(0)^{(m-p)/[(p+1)(m+1)]}\|u\|_{p+1}^{p+1} \quad (3.54)$$

for all sufficiently large t ($t \geq t_1$).

From Eqs. (3.51) and (3.54) yield

$$\begin{aligned} \dot{W}(t) &\geq ((1-\theta) - \zeta C(\varepsilon_1))H(t)^{-\theta}\dot{H}(t) \\ &+ \zeta \left[4\|u_t\|_2^2 + 4H(t) + \left(2\frac{p-1}{p+1} - \varepsilon_1 H(0)^{(m-p)/[(p+1)(m+1)]} \right) \|u\|_{p+1}^{p+1} \right] \end{aligned} \quad (3.55)$$

on the interval $[t_1, \infty)$.

We fix the constants ε_1 and ζ so that the inequalities

$$2\frac{p-1}{p+1} - \varepsilon_1 H(0)^{(m-p)/[(p+1)(m+1)]} > 0 \quad (3.56)$$

and

$$(1 - \theta) - \zeta C(\varepsilon_1) \geq 0 \quad (3.57)$$

are fulfilled.

So, we have

$$\dot{W}(t) \geq C[\|u_t\|_2^2 + H(t) + \|u\|_{p+1}^{p+1}]. \quad (3.58)$$

Finally, we get the blow up inequality

$$\dot{W}(t) \geq CW(t)^{1/(1-\theta)}, \quad (3.59)$$

valid on the interval $[t_2, \infty)$, for sufficiently large t_2 and appropriately diminished $\theta > 0$. Following the arguments from Theorem 2, it remains to estimate the term $(\int uu_t)^{1/(1-\theta)}$. Denote $1/(1-\alpha) = 1+v$, where $v > 0$. Then using the Holder and the Young inequalities we have

$$\left(\int uu_t\right)^{1+v} \leq C \left(\|u\|_{p+1}^{(1+v)\lambda} \left(\int_{B(L+t)} 1\right)^{(1+v)\lambda/r} + \|u_t\|_2^{(1+v)\mu}\right), \quad (3.60)$$

where $1/\mu + 1/\lambda = 1$ and

$$\frac{1}{2} = 1/(p+1) + 1/r. \quad (3.61)$$

We choose μ such that $(1+v)\mu = 2$. Then $\mu = 2/(1+v) < 2$, $\lambda > 2$ and if v is small, μ and λ are near 2. Therefore, we fix v so small that

$$2 < (1+v)\lambda < p+1. \quad (3.62)$$

Then, from Eq. (3.60), we have

$$\left(\int uu_t\right)^{1+v} \leq C \left(\|u\|_{p+1}^{p+1} \left(\int_{B(L+t)} 1\right)^{(1+v)\lambda/r} (\exp(C(L+t)^{-A_1+1}))^{-K} + \|u_t\|_2^2\right), \quad (3.63)$$

where $K = p+1 - (1+v)\lambda$ is a positive constant. From Eq. (3.63) we get

$$\left(\int uu_t\right)^{1+v} \leq C\|u\|_{p+1}^{p+1},$$

which is valid on the interval $[t_2, \infty)$ for t_2 sufficiently large. This completes the proof of the blow up inequality (3.59). \square

Since $H(t_2) > 0$, we can diminish additionally the constant ζ to insure also that $W(t_2) > 0$. Now, the integration of Eq. (3.59) completes the proof of Theorem 3. \square

Acknowledgements

This work was completed during the visit of the author at the Mathematical Department of the Chalmers University of Technology and the University of Goteborg in the period January–April 1997. The support and the kind hospitality there are gratefully acknowledged. The author would like to thank Professor Howard A. Levine for his helpful remarks on the final edition of the manuscript. The acknowledgements are also due to Professor Sang Ro Park for observing an important misprint in the manuscript.

References

- [1] J. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, *Quart. J. Math. Oxford* 28 (2) (1977) 473–486.
- [2] H. Brezis, *Analyse Fonctionnelle, Théorie et Applications*, Masson, Paris, 1983.
- [3] R.T. Glassey, Blow-up theorems for nonlinear wave equation, *Math. Z.* 132 (1973) 183–203.
- [4] V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, *J. Diff. Eq.* 109 (1994) 295–308.
- [5] A. Haraux, E. Zuazua, Decay estimates for some semilinear damped hyperbolic problems, *Arch. Rational Mech. Anal.* 100 (1988) 191–206.
- [6] R. Ikehata, Some remarks on the wave equations with nonlinear damping and source terms, *Nonlinear Anal.*, to appear.
- [7] M. Kopackova, Remarks on bounded solutions of a semilinear dissipative hyperbolic equation, *Comment. Math. Univ. Carolin.* 30 (4) (1989) 713–719.
- [8] H.A. Levine, Instability and nonexistence of global solutions of nonlinear wave equations of the form $Pu_{tt} = Au + F(u)$, *TAMS* 192 (1974) 1–21.
- [9] H.A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, *SIAM J. Math. Anal.* 5 (1974) 138–146.
- [10] H.A. Levine, J. Serrin, A global nonexistence theorem for quasilinear evolution equations with dissipation, *Arch. Rational Mech. Anal.* 137 (1997) 341–361.
- [11] H.A. Levine, S.R. Park, J. Serrin, Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation, *J. Math. Anal. Appl.*, to appear.
- [12] M. Ohta, Remarks on blowup of solutions for nonlinear evolution equations of second order, *Adv. in Math. Sc. and Appl.* 8 (1998) 901–910.
- [13] K. Ono, On global solutions and blow up solutions of nonlinear Kirchoff strings with nonlinear dissipations, *J. Math. Anal. Appl.* 216 (1997) 321–342.
- [14] L.E. Payne, D. Sattinger, Saddle points and instability on nonlinear hyperbolic equations, *Israel Math. J.* 22 (1981) 273–303.
- [15] I. Segal, Nonlinear semigroups, *Anal. Math.* 78 (1963) 339–364.
- [16] M. Tsutsumi, On solutions of semilinear differential equations in Hilbert space, *Math. Japan* 17 (1972) 173–193.