REGULARITY AND SCATTERING FOR THE WAVE EQUATION WITH A CRITICAL NONLINEAR DAMPING

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Abstract. We show that the nonlinear wave equation $\Box u + u_t^3 = 0$ is globally well-posed in radially symmetric Sobolev spaces $H^k_{\mathrm{rad}}(\mathbb{R}^3) \times H^{k-1}_{\mathrm{rad}}(\mathbb{R}^3)$ for all integers $k > 2$. This partially extends the well-posedness in $H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$ for all $k \in [1, 2]$, established by Lions and Strauss [12]. As a consequence we obtain the global existence of C^∞ solutions with radial C_0^∞ data. The regularity problem requires smoothing and non-concentration estimates in addition to standard energy estimates, since the cubic damping is critical when $k = 2$. We also establish scattering results for initial data $(u, u_t)|_{t=0}$ in radially symmetric Sobolev spaces.

1. Introduction

Let $\Box = \partial_t^2 - \Delta$ be the wave operator on $\mathbb{R} \times \mathbb{R}^3$ and $Du = (u_t, \nabla u)$ be the space-time derivative of u. We consider the wave equation with nonlinear damping

$$\Box u + u_t^3 = 0, \qquad x \in \mathbb{R}^3, \quad t > 0, \tag{1.1}$$

and initial conditions

$$u|_{t=0} = u_0, \qquad u_t|_{t=0} = u_1, \qquad x \in \mathbb{R}^3, \tag{1.2}$$

where $(u_0, u_1) \in H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$. Global well-posedness (existence and uniqueness) in Sobolev spaces is an open question for $k > 2$. Interestingly, the answer is affirmative for $k \in [1, 2]$ due to Lions and Strauss [12]; see also Joly, Metivier and Rauch [7] and Liang [11]. Below we outline the major difficulties to obtain well-posedness in Sobolev spaces with $k > 2$ and explain our partial solution. We use D^α to denote partial derivatives of order $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and $\|\cdot\|_p$ to denote the norm in $L^p(\mathbb{R}^3)$ for $p \in [1, \infty]$.

A priori estimates are essential for any global well-posedness result. Let us begin with a basic estimate in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ known as the energy inequality. From

$$\frac{d}{dt}\frac{\|Du(t)\|_2^2}{2} = -\|u_t(t)\|_4^4,$$

which is valid for sufficiently regular solutions of (1.1), we conclude that the energy decreases in time: $\|Du(t)\|_2 \le \|Du(0)\|_2$. Recall that we only multiply with u_t and apply the divergence theorem to obtain the above energy identity.

To establish higher regularity, we look for a priori estimates in $H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. Assuming that (1.1) can be differentiated once, we have

$$\Box D^\alpha u + 3u_t^2 D^\alpha u_t = 0, \qquad |\alpha| = 1.$$

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The above equation is also dissipative, since multiplying with $D^\alpha u_t$ and integrating on $\mathbb{R}^3$ yields the identity

$$\frac{d}{dt}\frac{\|DD^\alpha u(t)\|_2^2}{2} = -3\|u_t(t)D^\alpha u_t(t)\|_2^2.$$

Hence $\|DD^\alpha u(t)\|_2 \le \|DD^\alpha u(0)\|_2$ for $|\alpha| = 1$, i.e., all second-order norms decrease.

We can make such formal calculations rigorous using the monotonicity of nonlinear damping and standard approximation arguments. (Monotonicity means that $(u_t^3 - v_t^3)(u_t - v_t) \ge 0$ for any two functions. This property implies that the evolution governed by (1.1), (1.2) contracts initial data in the energy space; see Lemma 3.1 or [12], [7], [11].) Once we have suitable estimates and global well-posedness in $H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$ for $k = 1$ and $k = 2$, we can extend the result to all $k \in [1, 2]$ by interpolation.

Monotonicity is not sufficient, however, to show global well-posdness for $k > 2$. Returning to equation (1.1), we notice that high-order derivatives produce non-dissipative terms. In particular, the equation for $D^\alpha u$ with $|\alpha| = 2$ is

$$\Box D^\alpha u + 3u_t^2 D^\alpha u_t + \sum_{\beta+\gamma=\alpha} c_{\beta,\gamma} u_t D^\beta u_t D^\gamma u_t = 0,$$

where $c_{\beta,\gamma}$ are some constants. Now $\|DD^\alpha u(t)\|_2$ is not necessarily a decreasing function of t, as the energy identity is more complicated:

$$\begin{aligned} &\frac{d}{dt}\frac{\|DD^\alpha u(t)\|_2^2}{2} + \sum_{\beta+\gamma=\alpha} c_{\beta,\gamma} \int u_t D^\beta u_t D^\gamma u_t D^\alpha u_t \, dx \\ = \quad & -3\|u_t(t)D^\alpha u_t(t)\|_2^2. \end{aligned}$$

The integral is neither positive nor linearly bounded by the other terms, so it is not clear whether $\|DD^\alpha u(t)\|_2 < \infty$ at all t. To the best of our knowledge, the question is still open for data of arbitrary size in $H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$. We should mention that the global existence and asymptotic behavior for small data are well understood; see Klainerman and Ponce [9], Hörmander [6], and the recent work of Matsuyama [13] and references therein.

This paper establishes global well-posedness for large data under the assumption of radial symmetry. To state the result, we let $k \ge 1$ and introduce

$$H^k_{\mathrm{rad}}(\mathbb{R}^3) = \{u \in H^k(\mathbb{R}^3) : u \text{ is a function of } |x|\}.$$

Clearly $u(x)$ depends only on $|x|$ if and only if

$$(x_i\partial_{x_j} - x_j\partial_{x_i})u(x) = 0, \qquad 1 \le i < j \le 3.$$

Such spaces are invariant under the evolution map determined by (1.1), (1.2). The most important case is $k = 3$, since it ensures that $u_t, Du_t \in L^2(\mathbb{R}_+, L^\infty(\mathbb{R}^3))$. Then higher regularity follows from energy estimates and simple induction.

Theorem 1.1. *Assume that the initial data $(u_0, u_1) \in H^3_{\mathrm{rad}}(\mathbb{R}^3) \times H^2_{\mathrm{rad}}(\mathbb{R}^3)$. Then problem (1.1), (1.2) admits a unique global solution u, such that*

$$D^\alpha u \in C(\mathbb{R}_+, H^{3-|\alpha|}_{\mathrm{rad}}(\mathbb{R}^3)), \qquad |\alpha| \le 3.$$

Corollary 1.2. *Assume that the initial data* $(u_0, u_1) \in H^3_{\mathrm{rad}}(\mathbb{R}^3) \times H^2_{\mathrm{rad}}(\mathbb{R}^3)$. *Then the global solution of problem (1.1), (1.2) satisfies*

$$\begin{aligned} \sum_{1\le|\alpha|\le 2} \int_0^t \|D^\alpha u(s)\|_\infty^2\, ds \quad &\le \quad C_1(u_0,u_1), \qquad t \ge 0, \\ \sum_{|\alpha|\le 2} \|DD^\alpha u(t)\|_2 \quad &\le \quad C_2(u_0,u_1), \qquad t \ge 0, \end{aligned}$$

for implicit constants $C_k(u_0, u_1)$, $k = 1, 2$, *which are finite whenever the norm* $\|u_0\|_{H^3} + \|u_1\|_{H^2}$ *is finite; see the remark below.*

Remark 1.3. The implicit bounds come from Proposition 3.2 dealing with the non-concentration of certain space-time norms. Such results are typical for wave and Schrödinger equations with critical nonlinearities; see Struwe [19], Grillakis [5], Shatah and Struwe [16], Bahouri and Gérard [1], and Tao, Visan, and Zhang [23]. It may be possible to obtain explicit bounds by the induction on energy argument of Bourgain [2] and Tao [21], [22].

Theorem 1.4. *Assume that the initial data* $(u_0, u_1) \in H^k_{\mathrm{rad}}(\mathbb{R}^3) \times H^{k-1}_{\mathrm{rad}}(\mathbb{R}^3)$, *where* $k \ge 4$ *is an integer. Then problem (1.1), (1.2) admits a unique global solution* u, *such that*

$$D^\alpha u \in C(\mathbb{R}_+, H^{k-|\alpha|}_{\mathrm{rad}}(\mathbb{R}^3)), \qquad |\alpha| \le k.$$

If $(u_0, u_1) \in C^\infty_{\mathrm{rad}}(\mathbb{R}^3) \times C^\infty_{\mathrm{rad}}(\mathbb{R}^3)$ *and* (u_0, u_1) *have compact support, then problem (1.1), (1.2) admits a unique global solution* u *with compact support in* x, *such that*

$$u \in C^\infty(\mathbb{R}_+, C^\infty_{\mathrm{rad}}(\mathbb{R}^3)).$$

The proofs of Theorem 1.1, Corollaries 1.2 and Theorem 1.4 rely on the "forbidden" Strichartz estimate and non-concentration arguments. The $L^1_t L^2_x - L^2_t L^\infty_x$ Strichartz estimate (due to Klainerman and Machedon [8]) for the wave equation in $\mathbb{R} \times \mathbb{R}^3$ is valid only for radially symmetric solutions which explains the condition for radially symmetric data. The non-concentration of space-time norms

$$\int_I \|u_t(t) D u_t(t)\|_2^2\, dt \to 0, \qquad |I| \to 0. \tag{1.3}$$

is rather simple compared with the non-concentration argument for semilinear wave and Schrödinger equations in [19], [5], [16], [2], [21], [1], and [23]. Nevertheless, using (1.3) gives rise to the implicit constants in Corollary 1.2. To remove the condition for radial symmetry we may also need a more involved argument along the lines of Colliander, Keel, Staffilani, Takaoka, and Tao [3].

In conclusion, the critical nonlinear damping in $\mathbb{R}^3$ is quite different from other critical nonlinearities (at least for radial data). An indication is the invariant scaling of equation (1.1):

$$u(x,t) \mapsto u_L(x,t) = L^{1/2} u(x/L, t/L), \qquad L > 0.$$

We see that the kth-order norms scale as

$$\|D^\alpha u_L(t)\|_2 = L^{2-k} \|D^\alpha u(t/L)\|_2, \qquad |\alpha| = k,$$

so problem (1.1), (1.2) is supercritical for $k < 2$ and critical for $k = 2$. Surprisingly the global well-posedness for such k is already known from [12], where monotonicity plays a more decisive role than scaling invariance. This is the main difference between wave equations with nonlinear damping and other semilinear wave and Schrödinger equations.

For completeness we also study the long time behavior of solutions constructed in Theorem 1.1 and Corollary 1.2. It turns out that such solutions are asymptotically free, since the term u_t^3 is supercritical for scattering theory in $\mathbb{R}^3$. In comparison, the cubic damping is critical (and modifies the asymptotic profiles) in the following related equations:

$$\Box u + u + u_t^3 = 0, \qquad x \in \mathbb{R}, \quad t > 0,$$
$$\Box u + u_t^3 = 0, \qquad x \in \mathbb{R}^2, \quad t > 0.$$

Asymptotics for these are obtained by Delort [4] and Sunagawa [20] and by Kubo [10], respectively; see also Mochizuki [14] for the wave equation with general nonlinear damping and Nakanishi [15] for the Sobolev critical Klein-Gordon equation. In the case of equation (1.1) we readily verify the sufficient condition for scattering:

$$\int_0^\infty \|D^\alpha u_t^3(t)\|_2 \, dt < \infty, \qquad 0 \leq |\alpha| \leq 2,$$

where the initial data $(u_0, u_1) \in H^3_{\text{rad}}(\mathbb{R}^3) \times H^2_{\text{rad}}(\mathbb{R}^3)$. In fact, the above is a simple consequence of the estimates in Corollary 1.2.

Theorem 1.5. *Assume that the initial data* $(u_0, u_1) \in H^3_{\text{rad}}(\mathbb{R}^3) \times H^2_{\text{rad}}(\mathbb{R}^3)$. *The global solution u of problem (1.1), (1.2) is asymptotically free:*

$$\|D^\alpha u(t) - D^\alpha u_+(t)\|_2 \to 0, \qquad t \to \infty,$$

for $1 \leq |\alpha| \leq 3$, *where u_+ is a solution of $\Box u_+ = 0$ with initial data*

$$Du_+(x, 0) \in H^2_{\text{rad}}(\mathbb{R}^3) \times H^1_{\text{rad}}(\mathbb{R}^3).$$

Remark 1.6. We expect similar scattering results in $H^k_{\text{rad}}(\mathbb{R}^3) \times H^{k-1}_{\text{rad}}(\mathbb{R}^3)$ with integer $k \geq 4$.

The rest of this paper is organized as follows. Section 2 contains several basic facts and estimates for the wave equations in $\mathbb{R} \times \mathbb{R}^3$. The regularity problem in $H^3_{\text{rad}}(\mathbb{R}^3) \times H^2_{\text{rad}}(\mathbb{R}^3)$ is split between Sections 3 and 4. In Section 5 we use simple induction to prove Theorem 1.4 about well-posedness in $H^k_{\text{rad}}(\mathbb{R}^3) \times H^{k-1}_{\text{rad}}(\mathbb{R}^3)$ with $k > 3$. The scattering results of Theorem 1.5 are verified in the last Section 6.

2. Basic Estimates

The most important tools in this paper are the $L^1_t L^2_x - L^\infty_t H^1_x$ (energy) estimate and $L^1_t L^2_x - L^2_t L^\infty_x$ (radial Strichartz) estimate for the wave equation in $\mathbb{R} \times \mathbb{R}^3$.

Lemma 2.1. *Let u be a solution of the Cauchy problem in* $\mathbb{R} \times \mathbb{R}^3$

$$\Box u = F, \qquad u|_{t=0} = u_0, \qquad u_t|_{t=0} = u_1.$$

(a) For any source and initial data, u satisfies the energy estimate

$$\|Du(t)\|_2 \leq C(\|\nabla u_0\|_2 + \|u_1\|_2) + C \int_0^t \|F(s)\|_2 \, ds$$

with an absolute constant C for all $t \geq 0$.

(b) For radial source and initial data, u is also a radial function which satisfies

$$\left(\int_0^t \|u(s)\|_\infty^2 \, ds \right)^{1/2} \leq C(\|\nabla u_0\|_2 + \|u_1\|_2) + C \int_0^t \|F(s)\|_2 \, ds$$

with an absolute constant C for all $t \geq 0$.

Part (a) is a classical result presented, for instance, in the books of Strauss [18], Hörmander [6], and Shatah and Struwe [17]. Klainerman and Machedon [8] have found the homogeneous version of estimate (b) which implies the non-homogeneous estimate stated here.

We will work with classical solutions of problem (1.1), (1.2) whose local properties are well understood. The following is a collection of useful facts.

Lemma 2.2. *Assume that $k \geq 3$ and $(u_0, u_1) \in H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$.*

(a) There exists $T > 0$, such that problem (1.1), (1.2) has a unique solution u satisfying

$$D^\alpha u \in C([0,T], H^{k-|\alpha|}(\mathbb{R}^3)), \qquad |\alpha| \leq k.$$

Moreover, we have

$$\sup_{t\in[0,T]} \|D^\alpha u(t)\|_2 \leq C_k,$$

where T and C_k can be chosen to depend continuously on $\|u_0\|_{H^k} + \|u_1\|_{H^{k-1}}$.

(b) The continuation principle holds: if $T_* = T_*(u_0, u_1)$ is the supremum of all numbers T for which (a) holds, then either $T_* = \infty$ or

$$\sup_{t\in[0,T_*)} \|D^\alpha u(t)\|_2 = \infty$$

for some α with $|\alpha| \leq k$.

(c) If the data (u_0, u_1) are spherically symmetric, the solution (u, u_t) is also spherically symmetric.

Unfortunately, we do not have a single reference for all facts. The aforementioned books [18], [6], and [17] discuss these and other folklore results about nonlinear wave equations.

We conclude with an elementary functional inequality to replace the usual Gronwall inequality in some estimates.

Lemma 2.3. *Let $N : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function and $\delta > 0$ be a constant. If, for some non-decreasing function $A : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $B > 0$,*

$$N(t) \leq A(t) + BN(t - \delta)$$

holds for all $t \geq \delta$, then

$$N(t) \leq (1 + B)^{1+t/\delta}(N(0) + A(t)),$$

for all $t \geq \delta$.

Proof. If $t = \delta$ there is nothing to be proved. For $t > \delta$ there exists a positive integer n, such that $n\delta < t \leq (n + 1)\delta$. From the inequality for N, we obtain the chain of inequalities

$$\begin{aligned} N(n\delta) &\leq A(n\delta) + BN((n-1)\delta) \\ &\leq A(n\delta) + BA((n-1)\delta) + B^2N((n-2)\delta) \\ &\leq \cdots \\ &\leq (1 + B + \cdots + B^{n-1})A(n\delta) + B^nN(0). \end{aligned}$$

The final step uses $A(k\delta) \le A(n\delta)$ whenever $k \le n$. Bounding the geometric sequence of B by a simpler function, we can write

$$N(n\delta) \le (1+B)^{n-1} A(n\delta) + B^n N(0).$$

Since $n\delta < t \le (n+1)\delta$, we have a few additional steps:

$$\begin{aligned} N(t) &\le A(t) + BN(n\delta) \\ &\le A(t) + B(1+B)^{n-1}A(n\delta) + B^{n+1}N(0) \\ &\le (1+B)^n A(t) + (1+B)^{n+1} N(0). \end{aligned}$$

Increasing n to t/δ, we complete the proof. $\square$

3. Non-concentration and Smoothing Estimates

Lemma 2.2 (b) implies global well-posedness when all $\|D^\alpha u(t)\|_2$, $|\alpha| \le 3$, are locally bounded functions of t. Hence our goal is to show that such norms can not blow up in finite time.

We begin with two preliminary estimates for problem (1.1), (1.2) involving first and second order norms. These results, called the energy dissipation laws, are discussed in the introduction. We outline the proofs for completeness.

Lemma 3.1. *Assume that $(u_0, u_1) \in H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ and $|\alpha| = 1$. Let u be the solution of problem (1.1), (1.2) extended on a maximal interval $[0, T_*)$ by Lemma 2.2. (The main theorem claims $T_* = \infty$, but this result is not established yet.) Then*

$$\begin{aligned} \frac{1}{2}\|Du(t)\|_2^2 + \int_0^t \|u_s(s)\|_4^4\, ds &= \frac{1}{2}\|Du(0)\|_2^2, \\ \frac{1}{2}\|DD^\alpha u(t)\|_2^2 + 3\int_0^t \|u_s(s) D^\alpha u_s(s)\|_2^2\, ds &= \frac{1}{2}\|DD^\alpha u(0)\|_2^2, \end{aligned}$$

for $0 \le t < T_*$. Thus the following norms of u are bounded functions of t:

$$\|Du(t)\|_2 \le \|Du(0)\|_2, \quad \|DD^\alpha u(t)\|_2 \le \|DD^\alpha u(0)\|_2, \quad |\alpha| \le 1.$$

Proof. To show the first-order identity, we combine the divergence theorem and

$$0 = (\Box u + u_t^3) u_t = \left(\frac{|Du|^2}{2}\right)_t - \operatorname{div}(u_t \nabla u) + u_t^4.$$

The result follows from integration on $\mathbb{R}^3$ if $u(x,t)$ has compact support with respect to x. More generally, we can approximate $(u_0(x), u_1(x))$ with compactly supported C^∞ functions and use the finite propagation speed to show that the boundary integral of $\operatorname{div}(u_t \nabla u)$ is zero. Property (a) in Lemma 2.2 implies that the approximations will converge to the actual solution.

Similarly, we can differentiate equation (1.1) and multiply with $D^\alpha u_t(x,t)$ to show the second-order identity. Recall that we deal with solutions that have all third-order derivatives in $L^2(\mathbb{R}^3)$. $\square$

A simple corollary of Lemma 3.1 is the non-concentration of space-time norms arising from the damping.

Proposition 3.2. *Assume that $(u_0,u_1)\in H^3(\mathbb{R}^3)\times H^2(\mathbb{R}^3)$. Let u be the solution of problem (1.1), (1.2) extended on a maximal interval $[0,T_*)$ by Lemma 2.2. For every $\epsilon>0$ there exists $\delta>0$, such that*

$$\int_{t_1}^{t_2}\left(\|u_s(s)\|_4^4+\sum_{|\alpha|=1}\|u_s(s)D^\alpha u_s(s)\|_2^2\right)ds<\epsilon$$

whenever $[t_1,t_2]\subset[0,T_*)$ and $t_2-t_1<\delta$.

Proof. From Lemma 3.1, $\|u_s(s)\|_4^4+\|u_s(s)D^\alpha u_s(s)\|_2^2\in L^1([0,T_*))$ if $|\alpha|=1$. Let us define the measure

$$\mu(I)=\int_I\left(\|u_s(s)\|_4^4+\sum_{|\alpha|=1}\|u_s(s)D^\alpha u_s(s)\|_2^2\right)ds,\qquad I\subset[0,T_*),$$

where I can be any Borel set. Since μ is absolutely continuous with respect to the Lebesgue measure, we have $\mu(I)<\epsilon$ whenever $|I|<\delta$ is sufficiently small. $\square$

The next result combines Lemma 2.1 and Proposition 3.2 to bound the $L_t^2L_x^\infty$ norm of $D^\alpha u(x,t)$, $|\alpha|=1$. Here the radial symmetry of initial data is essential. Differentiating the main equation, we obtain

$$\Box D^\alpha u+3u_t^2D^\alpha u_t=0,\qquad |\alpha|=1.$$

Now Lemma 2.1 (b) implies that

$$\begin{aligned}\left(\int_0^t\|D^\alpha u(s)\|_\infty^2\,ds\right)^{1/2}&\\ \leq\ C\|DD^\alpha u(0)\|_2+C\int_0^t\|u_s^2(s)D^\alpha u_s(s)\|_2\,ds.&\end{aligned}\tag{3.1}$$

This is the starting point to establish the following estimate.

Proposition 3.3. *Assume that $(u_0,u_1)\in H^3_{\mathrm{rad}}(\mathbb{R}^3)\times H^2_{\mathrm{rad}}(\mathbb{R}^3)$. Let u be the local solution of problem (1.1), (1.2) constructed in Lemma 2.2. There exist an absolute constant C and positive constant $\delta=\delta(u_0,u_1)$, such that*

$$\begin{aligned}\sum_{|\alpha|=1}\left(\int_0^t\|D^\alpha u(s)\|_\infty^2\,ds\right)^{1/2}\quad&\leq\quad C\left(1+C\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2\right)^{1+t/\delta}\\&\qquad\times\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2\end{aligned}$$

for all $t\in[0,T_*)$.

Proof. Let $t\in(0,T_*)$. Using (3.1) and

$$\|u_s^2(s)D^\alpha u_s(s)\|_2\leq\|u_s(s)\|_\infty\|u_s(s)D^\alpha u_s(s)\|_2,$$

we can write

$$\begin{aligned}\left(\int_0^t\|D^\alpha u(s)\|_\infty^2\,ds\right)^{1/2}\quad&\leq\quad C\|DD^\alpha u(0)\|_2\\&\qquad+C\int_0^t\|u_s(s)\|_\infty\|u_s(s)D^\alpha u_s(s)\|_2\,ds.\end{aligned}$$

Summation over $|\alpha|=1$ yields the basic estimate

$$\begin{aligned}\sum_{|\alpha|=1}\left(\int_0^t \|D^\alpha u(s)\|_\infty^2\,ds\right)^{1/2} &\leq C\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2\\ &+C\int_0^t \|u_s(s)\|_\infty \sum_{|\alpha|=1}\|u_s(s)D^\alpha u_s(s)\|_2\,ds.\end{aligned}$$

We need the Cauchy inequality and non-concentration result of Proposition 3.2 to bound the derivatives of nonlinear damping in terms of the seminorm

$$N(t)=\sum_{|\alpha|=1}\left(\int_0^t \|D^\alpha u(s)\|_\infty^2\,ds\right)^{1/2}, \qquad t\geq 0.$$

Let $\delta>0$ be a small number to be determined later. For $t<\delta$, a direct application of the Cauchy inequality shows that

$$\begin{aligned}N(t) &\leq C\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2\\ &+C\left(\int_0^t \sum_{|\alpha|=1}\|u_s(s)D^\alpha u_s(s)\|_2^2\,ds\right)^{1/2} N(t).\end{aligned} \tag{3.2}$$

For $t\geq\delta$, the decomposition

$$\left(\int_0^{t-\delta}+\int_{t-\delta}^t\right)\|u_s(s)\|_\infty\|u_s(s)D^\alpha u_s(s)\|_2\,ds,$$

followed by two applications of the Cauchy inequality, results in

$$\begin{aligned}N(t) &\leq C\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2\\ &+CN(t-\delta)\left(\int_0^{t-\delta}\sum_{|\alpha|=1}\|u_s(s)D^\alpha u_s(s)\|_2^2\,ds\right)^{1/2}\\ &+CN(t)\left(\int_{t-\delta}^t\sum_{|\alpha|=1}\|u_s(s)D^\alpha u_s(s)\|_2^2\,ds\right)^{1/2}.\end{aligned}$$

To further simplify last estimate, we notice that

$$\left(\int_0^{t-\delta}\sum_{|\alpha|=1}\|u_s(s)D^\alpha u_s(s)\|_2^2\,ds\right)^{1/2}\leq C\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2,$$

which is a consequence of Lemma 3.1. Then

$$\begin{aligned}N(t) &\leq C\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2+C\left(\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2\right)N(t-\delta)\\ &+C\left(\int_{t-\delta}^t\sum_{|\alpha|=1}\|u_s(s)D^\alpha u_s(s)\|_2^2\,ds\right)^{1/2} N(t).\end{aligned} \tag{3.3}$$

We can now finish the proof using either (3.2) or (3.3). Recall that Proposition 3.2 yields $\delta > 0$ with the property

$$C\left(\int_{t_1}^{t_2}\sum_{|\alpha|=1}\|u_s(s)D^\alpha u_s(s)\|_2^2\,ds\right)^{1/2} < \frac{1}{2}$$

whenever $[t_1,t_2]\subset[0,T_*)$ and $t_2-t_1<\delta$. Here δ is a function of (u_0,u_1).

If $t<\delta$, we refer to estimate (3.2):

$$N(t)\le C\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2+\frac{1}{2}N(t).$$

This completes the proof for sufficiently small t.

If $t\ge\delta$, we go to estimate (3.3):

$$N(t)\le C\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2+C\left(\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2\right)N(t-\delta).$$

Applying Lemma 2.3, i.e., iterating the above inequality approximately t/δ times, we arrive at the final estimate for large t.

$$N(t)\le C\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2\left(1+C\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2\right)^{1+t/\delta},$$

with δ depending on (u_0,u_1). Unfortunately this dependence is not explicit. $\square$

The $L_t^2L_x^\infty$ norm of $D^\alpha u(x,t)$, $|\alpha|=2$, admits a similar estimate. Now we differentiate twice equation (1.1):

$$\Box D^\alpha u+3u_t^2D^\alpha u_t+\sum_{\beta+\gamma=\alpha}c_{\beta,\gamma}u_tD^\beta u_tD^\gamma u_t=0,$$

where $c_{\beta,\gamma}$ are constants. From Lemma 2.1 we obtain

$$\begin{aligned}\left(\int_0^t\|D^\alpha u(s)\|_\infty^2\,ds\right)^{1/2} &\le C\|DD^\alpha u(0)\|_2+C\int_0^t\|u_s^2(s)D^\alpha u_s(s)\|_2\,ds\\ &\quad +C\int_0^t\|u_s(s)(Du_s(s))^2\|_2\,ds.\end{aligned}\tag{3.4}$$

The rest is similar to Proposition 3.3 except that it involves third-order derivatives $D^\alpha u(x,t)$, $|\alpha|=3$. To handle such terms we need the energy estimate in Section 4.

Proposition 3.4. *Assume that $(u_0,u_1)\in H^3_{\rm rad}(\mathbb{R}^3)\times H^2_{\rm rad}(\mathbb{R}^3)$. Let u be the local solution of problem (1.1), (1.2) constructed in Lemma 2.2. There exist positive* *constants C and $\delta = \delta(u_0, u_1)$, such that*

$$\begin{aligned}
&\sum_{|\alpha|=2} \left(\int_0^t \|D^\alpha u(s)\|_\infty^2 \, ds \right)^{1/2} \\
\le\ & C \left(1 + C \sum_{|\alpha|=2} \|DD^\alpha u(0)\|_2 \right)^{1+t/\delta} \\
& \times \sum_{|\alpha|=2} \left(\|DD^\alpha u(0)\|_2 + \int_0^t \|u_s(s)\|_\infty^2 \|D^\alpha u_s(s)\|_2 \, ds \right)
\end{aligned}$$

for all $t \in [0, T_*)$.

Proof. Let $t \in (0, T_*)$ and $0 \le s \le t$. Substitute the upper bounds

$$\begin{aligned}
\|u_s^2(s) D^\alpha u_s(s)\|_2 &\le \|u_s(s)\|_\infty^2 \|D^\alpha u_s(s)\|_2, \\
\|u_s(s)(Du_s(s))^2\|_2 &\le \|Du_s(s)\|_\infty \|u_s(s) Du_s(s)\|_2,
\end{aligned} \tag{3.5}$$

into inequality (3.4). Then

$$\begin{aligned}
\left(\int_0^t \|D^\alpha u(s)\|_\infty^2 \, ds \right)^{1/2} &\le C \left(\|DD^\alpha u(0)\|_2 + \int_0^t \|u_s(s)\|_\infty^2 \|D^\alpha u_s(s)\|_2 \, ds \right) \\
&\quad + C \int_0^t \|Du_s(s)\|_\infty \|u_s(s) Du_s(s)\|_2 \, ds.
\end{aligned}$$

Summing over $|\alpha| = 2$ and introducing

$$N(t) = \sum_{|\alpha|=2} \left(\int_0^t \|D^\alpha u(s)\|_\infty^2 \, ds \right)^{1/2}, \qquad t \ge 0,$$

we have

$$\begin{aligned}
N(t) &\le C \sum_{|\alpha|=2} \left(\|DD^\alpha u(0)\|_2 + \int_0^t \|u_s(s)\|_\infty^2 \|D^\alpha u_s(s)\|_2 \, ds \right) \\
& C \int_0^t \|Du_s(s)\|_\infty \|u_s(s) Du_s(s)\|_2 \, ds.
\end{aligned}$$

Similarly to the previous proposition, we can estimated the derivatives of nonlinear damping in terms of $N(t)$. There are again two cases: $t < \delta$ and $t \ge \delta$, where $\delta > 0$ is chosen by the non-concentration result in Proposition 3.2.

For $t < \delta$, the use the Cauchy inequality to obtain

$$\begin{aligned}
N(t) &\le C \sum_{|\alpha|=2} \left(\|DD^\alpha u(0)\|_2 + \int_0^t \|u_s(s)\|_\infty^2 \|D^\alpha u_s(s)\|_2 \, ds \right) \\
&\quad + C \left(\int_0^t \|u_s(s) Du_s(s)\|_2^2 \, ds \right)^{1/2} N(t).
\end{aligned} \tag{3.6}$$

For $t \ge \delta$, we split the integral

$$\left(\int_0^{t-\delta} + \int_{t-\delta}^t \right) \|Du_s(s)\|_\infty \|u_s(s) Du_s(s)\|_2 \, ds$$

and apply the Cauchy inequality to each part. Then

$$\begin{aligned} N(t) &\le C\sum_{|\alpha|=2}\left(\|DD^\alpha u(0)\|_2+\int_0^t \|u_s(s)\|_\infty^2\|D^\alpha u_s(s)\|_2\,ds\right)\\ &\quad +C\left(\int_0^{t-\delta}\|u_s(s)Du_s(s)\|_2^2\,ds\right)^{1/2}N(t-\delta)\\ &\quad +C\left(\int_{t-\delta}^{t}\|u_s(s)Du_s(s)\|_2^2\,ds\right)^{1/2}N(t). \end{aligned}$$

It follows from Lemma 3.1 that

$$\left(\int_0^{t-\delta}\|u_s(s)Du_s(s)\|_2^2\,ds\right)^{1/2}\le C\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2,$$

so we have

$$\begin{aligned} N(t) &\le C\sum_{|\alpha|=2}\left(\|DD^\alpha u(0)\|_2+\int_0^t \|u_s(s)\|_\infty^2\|D^\alpha u_s(s)\|_2\,ds\right)\\ &\quad +C\left(\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2\right)N(t-\delta)\\ &\quad +C\left(\int_{t-\delta}^{t}\|u_s(s)Du_s(s)\|_2^2\,ds\right)^{1/2}N(t). \end{aligned} \tag{3.7}$$

Proposition 3.2 shows the existence of $\delta>0$, such that $[t_1,t_2]\subset[0,T_*)$ and $t_2-t_1<\delta$ imply

$$C\left(\int_{t_1}^{t_2}\|u_s(s)Du_s(s)\|_2^2\,ds\right)^{1/2}<\frac{1}{2}.$$

Hence estimates (3.6) and (3.7) yield

$$N(t)\le C\sum_{|\alpha|=2}\left(\|DD^\alpha u(0)\|_2+\int_0^t \|u_s(s)\|_\infty^2\|D^\alpha u_s(s)\|_2\,ds\right)$$

and

$$\begin{aligned} N(t) &\le C\sum_{|\alpha|=2}\left(\|DD^\alpha u(0)\|_2+\int_0^t \|u_s(s)\|_\infty^2\|D^\alpha u_s(s)\|_2\,ds\right)\\ &\quad +C\left(\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2\right)N(t-\delta), \end{aligned}$$

respectively. The proof is complete in the first case. Lemma 2.3 helps finish the proof in the second case. □

4. Energy Estimates and Global Existence of Radial Solutions in $H^3\times H^2$

In Section 3 we related smoothing estimates of $u(x,t)$ with estimates of $\|D^\alpha u(t)\|_2$, $1\le|\alpha|\le 3$. Here we obtain an inequality for the latter norms.

Proposition 4.1. *Assume that $(u_0,u_1)\in H^3_{\rm rad}(\mathbb{R}^3)\times H^2_{\rm rad}(\mathbb{R}^3)$ and let u be the local solution of problem (1.1), (1.2) constructed in Lemma 2.2. There exists an absolute constant C, such that*

$$\begin{aligned}\sum_{|\alpha|=2}\|DD^\alpha u(t)\|_2 \;&\le\; C\sum_{|\alpha|=2}\|DD^\alpha u(0)\|_2\\ &+C\int_0^t\|Du(s)\|_\infty^2\left(\sum_{|\alpha|=2}\|DD^\alpha u(s)\|_2\right)ds\\ &+C\left(\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2\right)\left(\int_0^t\|Du_s(s)\|_\infty^2\,ds\right)^{1/2}\end{aligned}$$

for all $t\in[0,T_*)$.

Proof. Differentiating (1.1) twice, we find that $D^\alpha u$ is a weak solution of

$$\Box D^\alpha u+3u_t^2D^\alpha u_t+\sum_{\beta+\gamma=\alpha}c_{\beta,\gamma}u_tD^\beta u_tD^\gamma u_t=0$$

with constant $c_{\beta,\gamma}$. Thus, $D^\alpha u$ satisfies the estimate in Lemma 2.1 (a):

$$\begin{aligned}\|DD^\alpha u(t)\|_2\;&\le\; C\|DD^\alpha u(0)\|_2\\ &+C\int_0^t\|u_s^2(s)D^\alpha u_s(s)\|_2\,ds\\ &+C\int_0^t\|u_s(s)(Du_s(s))^2\|_2\,ds\end{aligned}$$

for $t\in[0,T_*)$. Using (3.5) in both integrals and applying the Cauchy inequality to the second integral, we can write

$$\begin{aligned}\|DD^\alpha u(t)\|_2\;&\le\; C\|DD^\alpha u(0)\|_2\\ &+C\int_0^t\|Du(s)\|_\infty^2\|D^\alpha u_s(s)\|_2\,ds\\ &+C\left(\int_0^t\|Du_s(s)\|_\infty^2\,ds\right)^{1/2}\left(\int_0^t\|u_s(s)Du_s(s)\|_2^2\,ds\right)^{1/2}.\end{aligned}$$

To complete the proof, we add these estimates for all $|\alpha|=2$ and refer to Lemma 3.1 for the inequality

$$\int_0^t\|u_s(s)Du_s(s)\|_2^2\,ds\le C\sum_{|\alpha|=1}\|DD^\alpha u(0)\|_2^2.$$

□

We can finally show that $\|DD^\alpha u(t)\|_2$, with $|\alpha|\le 2$, do not blow up if the initial data have radial symmetry. Recall that such regularity is not preserved automatically, as the equation for second-order derivatives is not dissipative.

Proof of Theorem 1.1. It is clear from Lemma 3.1 that $\|D^\alpha u(t)\|_2$, $|\alpha|\le 2$, do not blow up in finite time. Thus we consider only third-order norms. Define

$$N(t)=\sum_{|\alpha|=2}\left(\|DD^\alpha u(t)\|_2^2+\int_0^t\|D^\alpha u(s)\|_\infty^2\,ds\right)^{1/2}.$$

We combine Propositions 3.4 and 4.1 to obtain

$$N(t) \le C_1(u_0,u_1,t)\left(N(0)+\int_0^t \|u_s(s)\|_\infty^2 N(s)\,ds\right)$$

for all $t \in [0,T_*)$, where $C_1(u_0,u_1,t)$ is a continuous increasing function of t determined from these results. Using the Gronwall inequality, we have an exponential estimate:

$$N(t) \le C_1(u_0,u_1,t)N(0)\exp\left(C_1(u_0,u_1,t)\int_0^t \|u_s(s)\|_\infty^2\,ds\right).$$

Proposition 3.3 shows that the integral of $\|u_s(s)\|_\infty^2$ is finite on any finite $[0,t]$. Hence $N(t)$ is finite on every finite subinterval of $[0,T_*)$. This completes the proof. □

Proof of Corollary 1.2. We start with the uniform estimates of $L_t^2L_x^\infty$ norms. Notice that inequality (3.1) is valid on any interval $[t_0,t]$ for all $|\alpha|=1$:

$$\begin{aligned}\sum_{|\alpha|=1}\left(\int_{t_0}^t \|D^\alpha u(s)\|_\infty^2\,ds\right)^{1/2} \quad &\le \quad C\sum_{|\alpha|=1}\|DD^\alpha u(t_0)\|_2 \\ &\quad +C\sum_{|\alpha|=1}\int_{t_0}^t \|u_s^2(s)D^\alpha u_s(s)\|_2^2\,ds.\end{aligned}$$

From the Cauchy inequality, we have

$$\begin{aligned}N_1(t) \quad &\le \quad C\sum_{|\alpha|=1}\|DD^\alpha u(t_0)\|_2 \\ &\quad +CN_1(t)\left(\int_{t_0}^t \|u_s(s)Du_s(s)\|_2^2\,ds\right)^{1/2},\end{aligned}$$

where

$$N_1(t)=\sum_{|\alpha|=1}\left(\int_{t_0}^t \|D^\alpha u(s)\|_\infty^2\,ds\right)^{1/2}, \qquad t \ge t_0.$$

Since the integral on $[0,\infty)$ is convergent, we can find a large t_0, such that

$$C\left(\int_{t_0}^t \|u_s(s)Du_s(s)\|_2^2\,ds\right)^{1/2} < \frac{1}{2}. \tag{4.1}$$

Thus $N_1(t) \le 2C\sum_{|\alpha|=1}\|DD^\alpha u(t_0)\|_2$ for all $t \ge t_0$. For bounded t the estimates follows from Theorem 1.1.

Using $N_1(t) \le C(u_0,u_1)$, $t \ge 0$, and Propositions 3.4 and 4.1, we can show that the remaining norms in Corollary 1.2 are also uniformly bounded. Let

$$N_2(t)=\sum_{|\alpha|=2}\left(\int_{t_0}^t \|D^\alpha u(s)\|_\infty^2\,ds\right)^{1/2}, \qquad t \ge t_0.$$

It follows from (3.4) that

$$\begin{aligned}\left(\int_{t_0}^t \|D^\alpha u(s)\|_\infty^2\,ds\right)^{1/2} \quad &\le \quad C\|DD^\alpha u(t_0)\|_2 + C\int_{t_0}^t \|u_s^2(s)D^\alpha u_s(s)\|_2\,ds \\ &\quad +C\int_{t_0}^t \|u_s(s)(Du_s(s))^2\|_2\,ds,\end{aligned}$$

where $|\alpha| = 2$. Thus

$$\begin{aligned} N_2(t) &\leq C \sum_{|\alpha|=2} \|DD^\alpha u(t_0)\|_2 \\ &\quad + C \int_{t_0}^{t} \|u_s(s)\|_\infty^2 \left(\sum_{|\alpha|=2} \|D^\alpha u_s(s)\|_2 \right) ds \\ &\quad + C N_2(t) \left(\int_{t_0}^{t} \|u_s(s) D u_s(s)\|_2^2 \, ds \right)^{1/2}. \end{aligned}$$

Now (4.1) shows that

$$\begin{aligned} N_2(t) &\leq 2C \sum_{|\alpha|=2} \|DD^\alpha u(t_0)\|_2 \\ &\quad + 2C \int_{t_0}^{t} \|u_s(s)\|_\infty^2 \left(\sum_{|\alpha|=2} \|D^\alpha u_s(s)\|_2 \right) ds \end{aligned}$$

for sufficiently large $t \geq t_0$. An immediate consequence is

$$\begin{aligned} N_2(t) &\leq 2C \sum_{|\alpha|=2} \|DD^\alpha u(t_0)\|_2 \\ &\quad + 2C N_3(t) \left(\int_{t_0}^{t} \|Du(s)\|_\infty^2 \, ds \right) \end{aligned} \tag{4.2}$$

with

$$N_3(t) = \sum_{|\alpha|=2} \sup_{s \in [t_0, t]} \|DD^\alpha u(s)\|_2, \qquad t \geq t_0.$$

Since Proposition 4.1 is also valid on $[t_0, t]$, we can write

$$\begin{aligned} N_3(t) &\leq C \sum_{|\alpha|=2} \|DD^\alpha u(t_0)\|_2 \\ &\quad + C N_3(t) \left(\int_{t_0}^{t} \|Du(s)\|_\infty^2 \, ds \right) \\ &\quad + C N_2(t) \left(\sum_{|\alpha|=1} \|DD^\alpha u(0)\|_2 \right) \end{aligned} \tag{4.3}$$

for all $t \geq t_0$. The convergence of

$$\int_0^\infty \|Du(s)\|_\infty^2 \, ds$$

and estimates (4.2), (4.3) are sufficient to bound uniformly $N_2(t) + N_3(t)$. □

5. Global Existence of Radial Solutions in $H^k \times H^{k-1}$, $k > 3$

Let u be the local solution of problem (1.1), (1.2) constructed in Lemma 2.2 and define the kth-order norms

$$E_k(t) = \sum_{|\alpha|=k} \|DD^\alpha u(t)\|_2.$$

We will use induction in $k \geq 2$ to show the global existence of u.

Proof of Theoreme 1.4. We already know that $E_2(t) < \infty$ for all $t \in [0, \infty)$; see Theorem 1.1. To prove that higher regularity is also preserved, we assume that $E_{k-1}(t) < \infty$ on $[0, \infty)$ but $E_k(t) < \infty$ only on $[0, T_*)$. If we apply D^α, $|\alpha| = k$, to equation (1.1), the result is

$$\Box D^\alpha u + 3u_t^2 D^\alpha u_t + \sum_{\mu+\nu+\lambda=\alpha} c_{\mu,\nu,\lambda} D^\mu u_t D^\nu u_t D^\lambda u_t = 0$$

with constants $c_{\mu,\nu,\lambda} \neq 0$ only when $\max(|\mu|, |\nu|,\ |\lambda|) \leq k-1$. Estimating $D^\alpha u$ by Lemma 2.1 (a), we have

$$\begin{aligned} \|DD^\alpha u(t)\|_2 \quad &\leq \quad C\|DD^\alpha u(0)\|_2 + C\int_0^t \|u_s^2(s) D^\alpha u_s(s)\|_2\, ds \\ &\quad + C \sum_{\mu+\nu+\lambda=\alpha} c_{\mu,\nu,\lambda} \int_0^t \|D^\mu u_s(s) D^\nu u_s(s) D^\lambda u_s(s)\|_2\, ds \end{aligned} \tag{5.1}$$

for $t \in [0, T_*)$. A key observation is that the second integral can be bounded in terms of $E_l(s)$, $0 \leq l \leq k-1$, and $\|D^\beta u(s)\|_\infty$, $1 \leq |\beta| \leq 2$.

If the largest index satisfies $|\mu| = k-1$, then $|\nu| + |\lambda| = 1$ and

$$\begin{aligned} \|D^\mu u_s(s) D^\nu u_s(s) D^\lambda u_s(s)\|_2 \quad &\leq \quad \|D^\mu u_s(s)\|_2 \|D^\nu u_s(s)\|_\infty \|D^\lambda u_s(s)\|_\infty \\ &\leq \quad E_{k-1}(s) \sum_{1\leq|\beta|\leq 2} \|D^\beta u(s)\|_\infty^2. \end{aligned}$$

If the indices satisfy $\max(|\mu|, |\nu|,\ |\lambda|) \leq k-2$, we combine the Hölder inequality and Sobolev embedding in $\mathbb{R}^3$ to obtain

$$\begin{aligned} \|D^\mu u_s(s) D^\nu u_s(s) D^\lambda u_s(s)\|_2 \quad &\leq \quad \|D^\mu u_s(s)\|_6 \|D^\nu u_s(s)\|_6 \|D^\lambda u_s(s)\|_6 \\ &\leq \quad C\|\nabla D^\mu u_s(s)\|_2 \|\nabla D^\nu u_s(s)\|_2 \|\nabla D^\lambda u_s(s)\|_2 \\ &\leq \quad C\sum_{l=1}^{k-1} E_l^3(s). \end{aligned}$$

Thus, (5.1) yields

$$\begin{aligned} \|DD^\alpha u(t)\|_2 \quad &\leq \quad C\|DD^\alpha u(0)\|_2 + C\int_0^t \|u_s^2(s) D^\alpha u_s(s)\|_2\, ds \\ &\quad + C\sum_{1\leq|\beta|\leq 2} \int_0^t E_{k-1}(s)\|D^\beta u(s)\|_\infty^2\, ds + C\sum_{l=1}^{k-1}\int_0^t E_l^3(s)\, ds \end{aligned}$$

for all $t \in [0, T_*)$. Adding these estimates for $|\alpha| = k$, we obtain

$$E_k(t) \leq CE_k(0) + C\int_0^t \|u_s(s)\|_\infty^2 E_k(s)\, ds + C_{k-1}(t),$$

where

$$C_{k-1}(t) = C\sum_{1\leq|\beta|\leq 2} \int_0^t E_{k-1}(s)\|D^\beta u(s)\|_\infty^2\, ds + C\sum_{l=1}^{k-1}\int_0^t E_l^3(s)\, ds$$

does not blow up in finite time due to the inductive assumption and Corollary 1.2. Now the Gronwall inequality and Proposition 3.3 imply that $E_k(t)$ is uniformly bounded on every finite subinterval of $[0, T_*)$. Hence u can be continued to a global solution with $E_k(t) < \infty$.

We have shown that, for all $k \geq 1$, $H^k \times H^{k-1}$ regularity is preserved during the evolution of radial data. Thus C^∞ regularity is also preserved during the evolution of compactly supported radial data. $\square$

6. Scattering of Radial Solutions in $H^3 \times H^2$

To study the behavior of u as $t \to \infty$, we restate Corollary 1.2:

$$\begin{aligned}
\sum_{|1 \leq \alpha| \leq 2} \int_0^\infty \|D^\alpha u(s)\|_\infty^2 \, ds &\leq C_1(u_0, u_1), \\
\sum_{|\alpha| \leq 2} \sup_{t \in [0,\infty)} \|DD^\alpha u(t)\|_2 &\leq C_2(u_0, u_1),
\end{aligned} \tag{6.1}$$

where $C_k(u_0, u_1)$, $k = 1, 2$, are finite constants if $\|u_0\|_{H^3} + \|u_1\|_{H^2}$ is finite.

Proof of Theorem 1.5. We rewrite problem (1.1), (1.2) in an equivalent integral form:

$$u(x,t) = u_l(x,t) - \int_0^t \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} \, u_s^3(s,x) \, ds.$$

Here u_l solves the homogeneous wave equation $\Box u_l = 0$ with the same initial data:

$$u_l(x,t) = \cos t\sqrt{-\Delta} \, u_0(x) + \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} \, u_1(x).$$

Define the asymptotic profile $u_+(x,t)$ as

$$u_+(x,t) = u_l(x,t) - \int_0^\infty \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} \, u_s^3(x,s) \, ds.$$

Clearly $\Box u_+ = 0$ and $w(x,t) = u(x,t) - u_+(x,t)$ is given by

$$w(x,t) = \int_t^\infty \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} \, u_s^3(x,s) \, ds.$$

Our goal is to show that $\|D^\alpha u_+(t)\|_2 < \infty$ and

$$\|D^\alpha w(t)\|_2 \to 0 \;\text{ as }\; t \to \infty, \qquad 1 \leq |\alpha| \leq 3.$$

Hence, it is sufficient to show that the following integrals converge:

$$\int_0^\infty \|D^\alpha u_s^3(s)\|_2 \, ds < \infty, \qquad |\alpha| \leq 2. \tag{6.2}$$

Using $|D^\alpha u_s^3| \leq 3u_s^2 |D^\alpha u_s|$, for $|\alpha| = 1$, and $|D^\alpha u_s^3| \leq 3u_s^2|D^\alpha u_s| + 6|u_s||Du_s|^2$, for $|\alpha| = 2$, we immediately obtain

$$\begin{aligned}
\int_0^\infty \|u_s^3(s)\|_2 \, ds &\leq C \int_0^\infty \|u_s(s)\|_\infty^2 \|u_s(s)\|_2 \, ds, \\
\int_0^\infty \|D^\alpha u_s^3(s)\|_2 \, ds &\leq C \int_0^\infty \|u_s(s)\|_\infty^2 \|D^\alpha u_s(s)\|_2 \, ds, \quad |\alpha| = 1, \\
\int_0^\infty \|D^\alpha u_s^3(s)\|_2 \, ds &\leq C \int_0^\infty \|u_s(s)\|_\infty^2 \|D^\alpha u_s(s)\|_2 \, ds \\
&\quad + C \int_0^\infty \|Du_s(s)\|_\infty^2 \|u_s(s)\|_2 \, ds, \quad |\alpha| = 2.
\end{aligned}$$

The integrals on the right sides converge due to (6.1). Hence these estimates imply the convergence in (6.2). $\square$

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