BLOW–UP FOR NONLINEAR DISSIPATIVE WAVE EQUATIONS IN $\mathbb{R}^n$

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ABSTRACT. We consider the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1}u_t = |u|^{p-1}u, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0,x) = u_0(x), \qquad u_t(0,x) = v_0(x) \end{cases}$$

for $1 \le m < p$, $p < n/(n-2)$ for $n \ge 3$. We prove that for any given numbers $\alpha > 0$, $\lambda \ge 0$ there exist infinitely many data u_0, v_0 in the energy space such that the initial energy $E(0) = \lambda$, the gradient norm $\|\nabla u_0\|_2 = \alpha$, and the solution of the above Cauchy problem blows up in finite time.

1. INTRODUCTION

We consider the Cauchy problem

(1)
$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1}u_t = |u|^{p-1}u, & \text{in } (0,\infty) \times \mathbb{R}^n \\ u(0,x) = u_0(x), \qquad u_t(0) = v_0(x), \end{cases}$$

where $1 \le m < p$, $p < n/(n-2)$ for $n \ge 3$. The data u_0, v_0 are compactly supported and in the energy space, that is $u_0 \in H^1(\mathbb{R}^n)$ and $v_0 \in L^2(\mathbb{R}^n)$.

The equation in (1) is a special case of the general nonlinearly damped wave equation

$$u_{tt} - \Delta u + g(t,x,u,u_t) = f(x,u)$$

with $g(t,x,u,u_t)u_u \ge 0$ and $f(x,u)u \ge 0$. Equations of this type arise naturally in many contexts – for instance in classical mechanics, fluid dynamics, quantum field theory (see [4], [17]). The interaction between two competitive forces, that is the nonlinear damping term $g(t,x,u,u_t)$ and the nonlinear source $f(x,u)$, makes the problem attractive from the mathematical point of view.

The Cauchy–Dirichlet and the Cauchy problem for equation (1) has been extensively studied in the last decade. See for example [2], [3], [5], [6], [7], [8], [9], [10], [11], [15], [18], [19], [21], [20], [24].

Global existence for the Cauchy problem (1) with arbitrarily chosen data in the energy space was proven in [21] for $p \le m$.

The case $m < p$ has a richer structure. In particular, it was shown (see [21] and [7]) that the solution of (1) blows up in finite time provided that the initial energy

$$E(0) = E(u_0, v_0) := \frac{1}{2}\|v_0\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{p+1}\|u_0\|_{p+1}^{p+1}$$

is sufficiently negative. Moreover, in [20] a potential well theory was established for a modified version of (1), and the exact decay of the solution was derived.
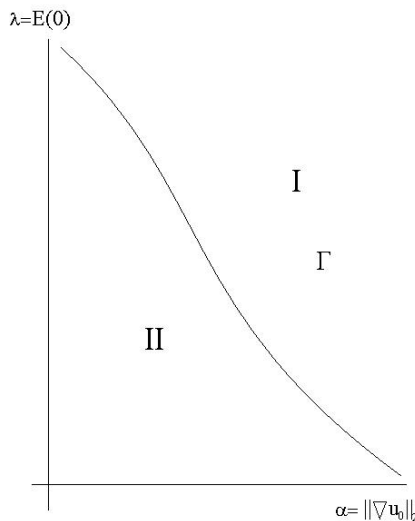
FIGURE 1

The negativity of initial energy was used to prove blow–up in many papers dealing with perturbed forms of the Cauchy problem (1). Here we shall mention only [1], [12], [14] and [16].

The case of *positive* initial energy for the problem (1) is much less understood. Only in the paper [10] does there appear a method to deal with this problem. There it was proven that, for any $\lambda \geq 0$, there are compactly supported data u_0, v_0 such that $E(0) = E(u_0, v_0) = \lambda$ and the solution of (1) with this data blows up in finite time. In order to control the decrease of the energy, the authors in [10] rescaled the data by setting

$$(2) \qquad\qquad u_0 = \sigma\psi, \qquad v_0 = \rho\phi.$$

The parameters ρ and σ are chosen in a convenient way and sufficiently large. Consequently, the authors were forced to consider a datum u_0 with sufficiently large $\|\nabla u_0\|_2$ (and large $\|u_0\|_\infty$, provided that $\psi \in L^\infty(\mathbb{R}^n)$) to get the blow–up result.

In view of this it is plausible to conjecture the existence of a curve (see Figure 1) Γ in the plane $(\|\nabla u_0\|_2, E(u_0, v_0))$ such that

(i) for any point (α, λ) on the right of this curve there are data u_0, v_0 such that $\|\nabla u_0\|_2 = \alpha$ and $E(u_0, v_0) = \lambda$ and the corresponding solution of (1) blows up in finite time (region I);

(ii) for any point (α, λ) on the left of this curve such a choice of the data cannot be made (region II).

In this paper we prove that this conjecture is incorrect, since such a curve does not exist, nor of course does the region II. In particular we prove that, for any given point $(\alpha, \lambda) \in (0, \infty) \times [0, \infty)$, there are infinitely many compactly supported data

u_0, v_0 in the energy space such that

$$\|\nabla u_0\|_2 = \alpha, \qquad E(u_0, v_0) = \lambda$$

and the solution blows up in finite time (Theorem 2).

We point out that for the Cauchy–Dirichlet problem associated with (1) an analogous result is false. Indeed it is well known that for $\|\nabla u_0\|_2$ and $E(u_0, v_0)$ sufficiently small the solution of the Cauchy–Dirichlet problem is global, due to the potential well theory (see [13]). On the contrary, the result in [10] is known (in a stronger form) for the Cauchy–Dirichlet problem, when the energy level is lower than the potential well level (see [11] and [24]).

Moreover, as a byproduct of the proof of our main result we get that, for $n \geq 3$, we can have blow–up for conveniently chosen data $u_0 \in H^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $v_0 \in L^2(\mathbb{R}^n)$ with arbitrarily small L^∞ norm and positive fixed energy λ.

We recall that for any point in a small neighborhood of the origin in the plane $(\|\nabla u_0\|_2, E(u_0, v_0))$ there are data small enough (namely data with σ and ρ sufficiently small in (2)), such that the solution of (1) is global. For this result in the case of linear damping ($m = 1$) and $p > 1 + 2/n$ see [23]. The result for nonlinear damping and sufficiently large p will be presented in [22]. So, in the above neighborhood we have two choices for the data: one that causes blow–up and the other that insures global existence. It is an interesting open problem whether this neighborhood is the only domain in the plane $(\|\nabla u_0\|_2, E(u_0, v_0))$ with this property.

The technique we use is a natural development of the arguments of [10]. However, some important differences arise. In particular, since we want to keep the norm $\|\nabla u_0\|_2$ fixed we cannot use the simple rescaling (2) of the data with $\sigma \to \infty$. Instead we use a transformation which keeps $\|\nabla u_0\|_2$ and $E(u_0, v_0)$ fixed, but the price for this is to raise the radius of the support of the data. As a consequence we cannot use the estimates in [10] because, roughly speaking, they are based on finite speed of propagation while for our purpose we need to use "pure"– $\mathbb{R}^n$ estimates. In this way we "clean" the estimates in [10] from dependence on the support of the data, or when we cannot avoid this dependence we take it into account explicitly. A further advantage of these estimates is that they may be adapted to other equations for which the finite speed of propagation property does not apply.

2. Preliminaries and statement of the main result

The following local existence and regularity result for the Cauchy problem (1) can be found in [21] (see also [2]).

Theorem 1. *Let*

$$1 \leq m < p, \qquad p < \frac{n}{n-2} \quad \text{for } n \geq 3.$$

Then, for any compactly supported data

$$u_0 \in H^1(\mathbb{R}^n), \qquad v_0 \in L^2(\mathbb{R}^n),$$

the Cauchy problem (1) *has a unique solution*

(3) $$u(t, x) \in C([0, T); H^1(\mathbb{R}^n)),$$

(4) $$u_t(t, x) \in C([0, T); L^2(\mathbb{R}^n)) \cap L^{m+1}([0, T) \times \mathbb{R}^n),$$

provided T is small enough.

To state our main result we introduce the domain D_n in $\mathbb{R}^2$ defined by the following inequalities

(5) $$m(n-2)(p+1)^2 < 4(p+n+1) \qquad \text{if } n \geq 3,$$

and

(6) $\quad p < \max \left\{ \dfrac{-(n-2)m^2 + 3(n+2)m}{(n-2)m^2 + (n-2)m + 4}, \quad \dfrac{-(n-2)m + 3n + 4}{(n-2)m + n} \right\} \qquad \text{if } n \geq 2,$

Namely

(7) $$D_n = \{(p,m) \in (1,\infty) \times [1,\infty) : (5) \text{ and } (6) \text{ hold}\}$$

when $n \geq 2$. In addition, when $n = 1$,

(8) $$D_1 = (1,\infty) \times [1,\infty).$$

Theorem 2. *Let*

(9) $$1 \leq m < p, \qquad p < \frac{n}{n-2} \quad \text{for } n \geq 3,$$

and $(p,m) \in D_n$. Then, for any $\lambda \geq 0$ and any $\alpha > 0$ there are infinitely many [1] compactly supported data $u_0 \in H^1(\mathbb{R}^n)$ and $v_0 \in L^2(\mathbb{R}^n)$ such that

(10) $$E(u_0, v_0) = \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\|v_0\|_2^2 - \frac{1}{p+1}\|u_0\|_{p+1}^{p+1} = \lambda,$$

(11) $$\|\nabla u_0\|_2 = \alpha$$

and the solution of the Cauchy problem (1) blows up in finite time, i.e. there is $T_{max} < \infty$ such that $\lim\limits_{t \to T_{max}^-} \|u(t)\|_{p+1} = \infty$.

As a byproduct of the proof of Theorem 2 we have

Corollary 1. *Let the assumptions of Theorem 2 hold and $n \geq 3$. For any $\alpha > 0$ and $\lambda \geq 0$ there are compactly supported data $u_0 \in H^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $v_0 \in L^2(\mathbb{R}^n)$, with $\|u_0\|_\infty$ arbitrarily small, such that (10)–(11) hold and the solution of the Cauchy problem (1) blows up in finite time.*

Remark 1. The technical condition $(p,m) \in D_n$ cuts a small part B_n from the natural set for the exponents of nonlinearities endowed by (9). How small is this cut is illustrated in Figure 2. The figure shows that when the space dimension n increases the set B_n becomes negligible. The shape of D_1 together with the behavior of B_n for large n, naturally leads to the conjecture that conditions (5)–(6) are method driven for all $n \in \mathbb{N}$.

In the proof we shall use the well known Gagliardo–Nirenberg estimate. For the sake of completeness we recall it here.

Lemma 1 (Gagliardo–Nirenberg)**.** *Let $1 \leq r < q \leq \infty$, and $q \geq 2$. Then the inequality*

$$\|v\|_q \leq C\|\nabla v\|_2^\theta \|v\|_r^{1-\theta}, \qquad \text{for } v \in H^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$$
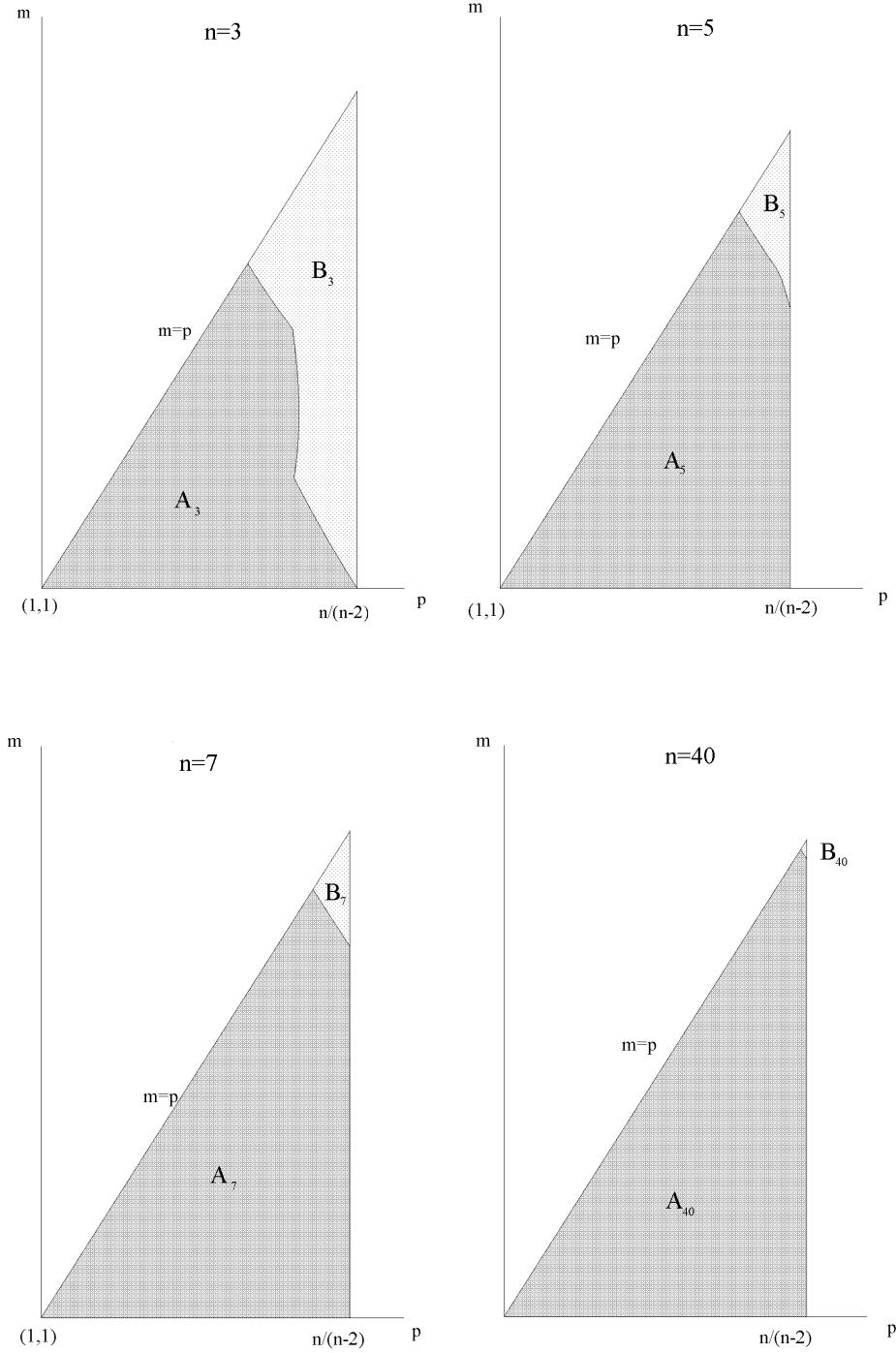
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[1]actually a continuous branch

FIGURE 2

$$A_n := \{(p,m) \in \mathbb{R}^2 : (9) \text{ holds and } (p,m) \in D_n\},$$

$$B_n := \{(p,m) \in \mathbb{R}^2 : (9) \text{ holds and } (p,m) \notin D_n\}.$$

It can be checked that for $n \geq 7$ only the inequalities (5) and (9) describe A_n while for $4 \leq n \leq 6$ the inequality (6) can be written in the simpler form

$$p[(n-2)m^2 + (n-2)m + 4] < -(n-2)m^2 + 3(n+2)m.$$

holds with some positive constant $C = C(r, q, n)$ and

$$(12) \qquad \theta = \frac{\dfrac{1}{r} - \dfrac{1}{q}}{\dfrac{1}{n} + \dfrac{1}{r} - \dfrac{1}{2}},$$

provided that $0 < \theta < 1$.

3. Proofs

Proof of Theorem 2. Let $\lambda \geq 0$ and $\alpha > 0$ be fixed, and let u_0 and v_0 be data to be chosen in the sequel, $u_0 \in H^1(\mathbb{R}^n)$, $v_0 \in L^2(\mathbb{R}^n)$, with support contained in a ball $B_L := \{x \in \mathbb{R}^n : |x| \leq L\}$, $L > 0$, such that $E(u_0, v_0) = \lambda$ and $\|\nabla u_0\|_2 = \alpha$. Let u be the corresponding solution of (1).

In the sequel C will indicate different positive constants, dependent only on p, m and n, but *independent* of L and u_0, v_0.

The energy of the solution u is

$$(13) \qquad E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}.$$

We recall one of the blow–up results in [21], [7]. There is a critical level of the energy E_{cr}, with

$$(14) \qquad -\max\{1, CL^{-C}\} \leq E_{\mathrm{cr}} \leq 0,$$

such that if

$$(15) \qquad E(\bar{t}) < E_{\mathrm{cr}}$$

for some $\bar{t} > 0$ then the solution of the Cauchy problem (1) blows–up in finite time.[2]

By the energy identity we have

$$(16) \qquad E(0) - E(t) = \int_0^t \|u_t\|_{m+1}^{m+1}.$$

Then the blow–up sufficient condition (15) can be rewritten as

$$(17) \qquad \int_0^{\bar{t}} \|u_t\|_{m+1}^{m+1} > E(0) - E_{\mathrm{cr}},$$

for some $\bar{t} > 0$.

The idea is to control the dissipation $\int_0^t \|u_t\|_{m+1}^{m+1}$ of the energy $E(t)$ in order to assure that the energy $E(t)$ overpasses the critical level E_{cr} for conveniently chosen data u_0, v_0 with prescribed energy $E(0)$ and $\|\nabla u_0\|$. All estimates we derive to control this dissipation hold locally, i.e. for a small time interval. A delicate point is to show that the data can be chosen such that in this small time interval there is a time $\bar{t}$ such that the blow–up condition (17) holds.

For contradiction we assume that the solution u is global, so that

$$(18) \qquad E(t) = \widetilde{E}(t) - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \geq E_{\mathrm{cr}}, \qquad \text{for all } t \geq 0,$$

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[2]Let us notice that the condition $\int_{\mathbb{R}^n} u_0 v_0 \geq 0$ mentioned in [21] can be avoided. Indeed, the requirement $\dot{F}(0) \geq 0$ at [21, p. 898 – Case 2], which leads to the condition mentioned above, can be skipped by taking ε so small that $\varepsilon|\dot{F}(0)| \leq L^A H(0)^{1-\alpha}/2$.

where

$$(19) \qquad \widetilde{E}(t) := \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2.$$

Namely,

$$(20) \qquad \|u(t)\|_{p+1} \le (p+1)^{1/(p+1)}[\widetilde{E}(t) - E_{\mathrm{cr}}]^{1/(p+1)} \qquad \text{for all } t \ge 0.$$

By using the Hőlder inequality and the finite speed of propagation of u, we have

$$(21) \qquad \|u_t\|_{m+1}^{m+1} \ge C(L+t)^{-n(m-1)/2}\|u_t\|_2^{m+1}$$

which becomes an identity when $m = 1$. To estimate $\|u_t\|_2^{m+1}$ from below we use the inequality

$$(22) \qquad \|g\|_2 \ge \|g\|_{H^{-1}(\mathbb{R}^n)}, \qquad \text{for all } g \in L^2(\mathbb{R}^n).$$

Indeed $H^{-1}(\mathbb{R}^n)$ (in the sequel simply denoted by H^{-1}) is a natural space for wave equations with data in the energy space.

A main tool in the proof is to decompose the solution u as the sum of the solution of the free wave equation with the same data and a remainder term, i.e.

$$(23) \qquad u = w + v,$$

where

$$(24) \qquad \begin{cases} w_{tt} - \Delta w = 0, & \text{in } (0,\infty) \times \mathbb{R}^n, \\ w(0,x) = u_0(x), \qquad w_t(0,x) = v_0(x), & \text{in } \mathbb{R}^n. \end{cases}$$

The remainder term v is a solution of the Cauchy problem

$$(25) \qquad \begin{cases} v_{tt} - \Delta v = F(t,x), & \text{in } (0,\infty) \times \mathbb{R}^n, \\ v(0,x) = 0, \qquad v_t(0,x) = 0, & \text{in } \mathbb{R}^n, \end{cases}$$

where

$$(26) \qquad F(t,x) := |u|^{p-1}u - |u_t|^{m-1}u_t.$$

The main difference with the corresponding proof in [10] is the following. In view of our further choice of the data in which the radius of the support L will *not be fixed*, we need the estimates to be either independent of L or, if that is not possible, to control carefully the L dependence. Actually we can avoid any L–dependence everywhere except in the next estimate.

By using the decomposition (23) together with (21) and (22) we have

$$(27) \qquad \begin{aligned} \|u_t\|_{m+1}^{m+1} &\ge C(L+t)^{-n(m-1)/2}\|u_t\|_{H^{-1}}^{m+1} \\ &\ge C(L+t)^{-n(m-1)/2}\left(\|w_t\|_{H^{-1}}^{m+1} - \|v_t\|_{H^{-1}}^{m+1}\right). \end{aligned}$$

By (27) we need estimates of the term $\|w_t\|_{H^{-1}}^{m+1}$ from below, and $\|v_t\|_{H^{-1}}^{m+1}$ from above, which hold for a small time interval.

To estimate $\|w_t\|_{H^{-1}}^{m+1}$ from below we use the fundamental solution of the free wave equation (24)

$$(28) \qquad \widehat{w}(t,\xi) = \cos(t|\xi|)\widehat{u_0}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{v_0}(\xi)$$

where $\widehat{w}$ is the Fourier transform of w. Then the arguments of [10] show that the inequality [10, 3.16] holds. In order to avoid the L–dependence in the estimate for $\|w_t\|_{H^{-1}}^{m+1}$, we can rewrite this inequality in the form

$$|\widehat{w}_t(t,\xi)|^2 \geq \frac{1}{2}|\widehat{v_0}|^2 - t^2|\xi|^2 \left(|\widehat{\nabla u_0}(\xi)|^2 + |\widehat{v_0}|^2\right).$$

This directly implies

$$\frac{1}{2}\|v_0\|_{H^{-1}}^2 \leq \|w_t\|_{H^{-1}}^2 + 2t^2\widetilde{E}(0),$$

which leads to the final L–independent estimate for $\|w_t\|_{H^{-1}}^{m+1}$

(29) $$\|w_t\|_{H^{-1}}^{m+1} \geq C\left(\|v_0\|_{H^{-1}}^{m+1} - Ct^{m+1}\widetilde{E}^{(m+1)/2}(0)\right).$$

The estimate of the term $\|v_t\|_{H^{-1}}^{m+1}$ from above is much more delicate. Taking the H^{-1} scalar product of both sides of (25) by v_t, using the identity

$$-(\Delta v, v_t)_{H^{-1}} = \frac{1}{2}\frac{d}{dt}\|\nabla v\|_{H^{-1}}^2,$$

and integrating on $[0,t]$ we obtain

$$\frac{1}{2}\|v_t\|_{H^{-1}}^2 + \frac{1}{2}\|\nabla v\|_{H^{-1}}^2 \leq \int_0^t \|F(\cdot,t)\|_{H^{-1}}\|v_t\|_{H^{-1}}.$$

In the above we use that the initial data in (25) are zero. Applying a Gronwall type inequality we obtain

(30) $$\|v_t\|_{H^{-1}} \leq \int_0^t \|F(\cdot,\eta)\|_{H^{-1}}\,d\eta.$$

Using (26) we have

(31) $$\|F(\cdot,\eta)\|_{H^{-1}} \leq \||u|^{p-1}u\|_{H^{-1}} + \||u_t|^{m-1}u_t\|_{H^{-1}}.$$

The Sobolev embedding states that

(32) $$\|g\|_{H^{-1}} \leq C(l,n)\|g\|_l, \qquad \text{for all } g \in L^l(\mathbb{R}^n),$$

where $C(l,n)$ is a positive constant, for any $1 < l \leq 2$ and $l \geq 2n/(n+2)$. We use (32) to estimate the first and the second term on the right–hand side of (31), with $l = (p+1)/p$ and $l = (m+1)/m$ respectively. For this choice of l the inequality (32) holds because $l \geq 2n/(n+2)$ since $1 \leq m < p < n/(n-2)$. Then we have

(33)
$$\|F(\cdot,\eta)\|_{H^{-1}} \leq C\left(\|u\|_{p+1}^p + \|u_t\|_{m+1}^m\right)$$
$$\leq C\left[(\widetilde{E}(\eta) - E_{\mathrm{cr}})^{p/(p+1)} + \|u_t\|_{m+1}^m\right],$$

where we have used the estimate (20).

The estimates (30) and (33) together with the Hőlder inequality with respect to t imply

$$\|v_t\|_{H^{-1}} \leq C\left[\int_0^t (\widetilde{E}(\eta) - E_{\mathrm{cr}})^{p/(p+1)}\,d\eta + t^{1/(m+1)}\left(\int_0^t \|u_t\|_{m+1}^{m+1}\right)^{m/(m+1)}\right].$$

By (16) and (18) we come to

(34) $$\|v_t\|_{H^{-1}} \leq C\left[\int_0^t (\widetilde{E}(\eta) - E_{\mathrm{cr}})^{p/(p+1)}\,d\eta + t^{1/(m+1)}(E(0) - E_{\mathrm{cr}})^{m/(m+1)}\right].$$

The advantage of this estimate compared with [10, 3.22] is that (34) is support independent while [10, 3.22] is not.

It is clear now that we need a local (in time) estimate for $\widetilde{E}(t)$ from above. Note that, by (19) and the energy identity,

$$(35) \qquad \widetilde{E}'(t) = -\|u_t\|_{m+1}^{m+1} + \int_{\mathbb{R}^n} |u|^{p-1} u u_t.$$

¿From (35) we shall derive an ordinary differential inequality of the type

$$\left(\widetilde{E}(t) + C\right)' \leq C_1 \left(\widetilde{E}(t) + C\right)^{\alpha}.$$

This can be done in two different ways and the best choice between them relies on minimizing the exponent α. This maximizes the time interval for which the final estimate applies. This optimal choice depends on the exponents p and m of the nonlinearities and on the dimension of the space n.

First estimate of $\widetilde{E}'(t)$. By using Young's inequality from (35) we get

$$(36) \qquad \begin{aligned} \widetilde{E}'(t) \leq & -\|u_t\|_{m+1}^{m+1} + \frac{m}{m+1} \int_{\mathbb{R}^n} |u|^{p(m+1)/m} + \frac{1}{m+1} \int_{\mathbb{R}^n} |u_t|^{m+1} \\ \leq & \frac{m}{m+1} \|u\|_{p(m+1)/m}^{p(m+1)/m}. \end{aligned}$$

To estimate the right–hand side of (36) we use the Gagliardo – Nirenberg estimate (Lemma 1) with

$$q = \frac{p(m+1)}{m} \qquad \text{and} \quad r = p+1.$$

Thus we have

$$(37) \qquad \widetilde{E}'(t) \leq C \left(\|\nabla u\|_2^{\theta_1} \|u\|_{p+1}^{1-\theta_1}\right)^{p(m+1)/m},$$

where

$$\theta_1 := \frac{\dfrac{1}{p+1} - \dfrac{m}{p(m+1)}}{\dfrac{1}{n} + \dfrac{1}{p+1} - \dfrac{1}{2}}.$$

An elementary calculation shows that

$$(38) \qquad \theta_1 = \frac{2n(p-m)}{p(m+1)[2n - (n-2)(p+1)]},$$

which gives $0 < \theta_1 < 1$ since $1 \leq m < p$ and $p < n/(n-2)$ (for $n \geq 3$).

By using estimates (19), (20) and (37) we obtain the inequality

$$\widetilde{E}'(t) \leq C \left[\widetilde{E}^{\theta_1/2}(t) \left(\widetilde{E}(t) - E_{\mathrm{cr}}\right)^{(1-\theta_1)/(p+1)}\right]^{p(m+1)/m}.$$

Since $E_{\mathrm{cr}} \leq 0$, it follows that

$$(39) \qquad \widetilde{E}'(t) \leq C \left(\widetilde{E}(t) - E_{\mathrm{cr}}\right)^{\gamma_1},$$

where

$$(40) \qquad \gamma_1 = \frac{p(m+1)}{m} \left(\frac{\theta_1}{2} + \frac{1-\theta_1}{p+1}\right).$$

Using (38) and (40), elementary calculations show that

$$(41) \qquad \gamma_1 = 1 + \frac{2(p-m)}{m[2n - (n-2)(p+1)]} > 1.$$

This concludes our first estimate of $\widetilde{E}'(t)$.

Second estimate of $\widetilde{E}'(t)$. Neglecting the first term in the right–hand side of (35) and using Schwarz inequality, we obtain

$$(42) \qquad \widetilde{E}'(t) \leq \int_{\mathbb{R}^n} |u|^{p-1} u u_t \leq \int_{\mathbb{R}^n} |u|^p |u_t| \leq \|u_t\|_2 \|u\|_{2p}^p.$$

We apply the Gagliardo–Nirenberg inequality (Lemma 1), with

$$q = 2p \qquad \text{and} \quad r = p+1,$$

to get

$$(43) \qquad \|u\|_{2p} \leq C \|\nabla u\|_2^{\theta_2} \|u\|_{p+1}^{1-\theta_2},$$

where

$$\theta_2 = \frac{\dfrac{1}{p+1} - \dfrac{1}{2p}}{\dfrac{1}{n} + \dfrac{1}{p+1} - \dfrac{1}{2}}.$$

An elementary calculation shows that

$$(44) \qquad \theta_2 = \frac{n(p-1)}{p[2n - (n-2)(p+1)]},$$

and $0 < \theta_2 < 1$ since $1 \leq m < p$ and $p < n/(n-2)$ (for $n \geq 3$). Then, since $E_{cr} \leq 0$, by (19), (20) and (43) we have

$$(45) \qquad \begin{aligned} \|u\|_{2p} &\leq C \widetilde{E}^{\theta_2/2}(t) \left[\widetilde{E}(t) - E_{cr} \right]^{(1-\theta_2)/(p+1)} \\ &\leq C \left(\widetilde{E}(t) - E_{cr} \right)^{(1-\theta_2)/(p+1) + \theta_2/2}. \end{aligned}$$

By using the estimates (19), (45) and the fact that $E_{cr} \leq 0$ we can rewrite (42) in the following form

$$(46) \qquad \widetilde{E}'(t) \leq C \left(\widetilde{E}(t) - E_{cr} \right)^{\gamma_2},$$

where

$$(47) \qquad \gamma_2 = \frac{1}{2} + p \left(\frac{\theta_2}{2} + \frac{1 - \theta_2}{p+1} \right).$$

Moreover, elementary calculations show that

$$(48) \qquad \gamma_2 = 1 + \frac{p-1}{2n - (n-2)(p+1)} > 1,$$

where we have used (44) as well. This concludes our second estimate of $\widetilde{E}'(t)$.

To choose between estimates (39) and (46) we minimize the exponent of $\left(\widetilde{E}(t) - E_{cr} \right)$. We set

$$(49) \qquad \gamma_0 := \min\{\gamma_1, \gamma_2\}.$$

By (41) and (48) it is clear that $\gamma_0 > 1$. So we can rewrite (39) and (46) as

$$(50) \qquad (\widetilde{E} - E_{\mathrm{cr}})'(t) = \widetilde{E}'(t) \leq C(\widetilde{E}(t) - E_{\mathrm{cr}})^{\gamma_0}.$$

Integrating (50) we get

$$(51) \qquad (\widetilde{E} - E_{\mathrm{cr}})^{1-\gamma_0}(t) \geq (\widetilde{E} - E_{\mathrm{cr}})^{1-\gamma_0}(0) - (\gamma_0 - 1)Ct.$$

For t small enough, namely

$$(52) \qquad t \leq T_1 := C\left(\widetilde{E}(0) - E_{\mathrm{cr}}\right)^{1-\gamma_0},$$

by (51) we have

$$(53) \qquad \widetilde{E}(t) - E_{\mathrm{cr}} \leq 2(\widetilde{E}(0) - E_{\mathrm{cr}}).$$

In the last step we used that $\gamma_0 > 1$. The estimate (53) is the final estimate for $\widetilde{E}(t)$, which applies for all t satisfying (52).

Turning back to the main estimate (34) and using the inequality (53), we obtain

$$\|v_t\|_{H^{-1}} \leq C\left[t(\widetilde{E}(0) - E_{\mathrm{cr}})^{p/(p+1)} + t^{1/(m+1)}\left(E(0) - E_{\mathrm{cr}}\right)^{m/(m+1)}\right],$$

as long as $t \leq T_1$. The above estimate yields

$$(54) \qquad \|v_t\|_{H^{-1}}^{m+1} \leq C\left[t^{m+1}(\widetilde{E}(0) - E_{\mathrm{cr}})^{p(m+1)/(p+1)} + t\left(E(0) - E_{\mathrm{cr}}\right)^m\right].$$

This is our final estimate of $\|v_t\|_{H^{-1}}^{m+1}$, which applies for $t \leq T_1$.

This estimate together with (29) gives us the possibility to rewrite the inequality (27) in the form

$$(55) \quad \|u_t\|_{m+1}^{m+1} \geq C(L+t)^{-n(m-1)/2}\left[\|v_0\|_{H^{-1}}^{m+1} - Ct^{m+1}\widetilde{E}^{(m+1)/2}(0)\right.$$
$$\left. -Ct^{m+1}(\widetilde{E}(0) - E_{\mathrm{cr}})^{p(m+1)/(p+1)} - Ct\left(E(0) - E_{\mathrm{cr}}\right)^m\right],$$

as long as $t \leq T_1$. Notice that the inequality (55) is valid also for $m = 1$, in which case the term $(L+t)^{-n(m-1)/2}$ reduces to 1 and the remainder of the proof works equally well. We shall give more details for the case $m = 1$ in the sequel. We simplify the inequality (55) in the following way

$$(56) \quad \|u_t\|_{m+1}^{m+1} \geq CL^{-n(m-1)/2}\left[\|v_0\|_{H^{-1}}^{m+1} - Ct^{m+1}(\widetilde{E}(0) - E_{\mathrm{cr}})^{p(m+1)/(p+1)}\right.$$
$$\left. -Ct\left(E(0) - E_{\mathrm{cr}}\right)^m\right],$$

as long as $t \leq \min\{L, T_1\}$; here we have taken into account that $\widetilde{E}(0) \leq \widetilde{E}(0) - E_{\mathrm{cr}}$, $p(m+1)/(p+1) > (m+1)/2$ together with the assumption $\widetilde{E}(0) > 1$. The last assumption is natural since in the sequel we shall choose data with sufficiently large $\widetilde{E}(0)$. Integrating (56) on $[0, t]$, $t \leq \min\{L, T_1\}$, we get

$$(57) \quad \int_0^t \|u_t\|_{m+1}^{m+1} \geq CL^{-n(m-1)/2}t\left[\|v_0\|_{H^{-1}}^{m+1} - Ct^{m+1}(\widetilde{E}(0) - E_{\mathrm{cr}})^{p(m+1)/(p+1)}\right.$$
$$\left. -Ct\left(E(0) - E_{\mathrm{cr}}\right)^m\right].$$

We recall that the inequality (17) assures the blow–up of the solution of (1). Taking into account the estimate (57), we have to prove that there exist data such that

$$
\text{(58)} \quad \int_0^t \|u_t\|_{m+1}^{m+1} \geq CL^{-n(m-1)/2}t\Big[\|v_0\|_{H^{-1}}^{m+1} - Ct^{m+1}(\widetilde{E}(0) - E_{\mathrm{cr}})^{p(m+1)/(p+1)}
$$
$$
-Ct\,(E(0) - E_{\mathrm{cr}})^m\Big] \geq E(0) - E_{\mathrm{cr}}
$$

for some $t \leq \min\{L, T_1\}$.

Now we start with the choice of the data u_0 and v_0. Let $\psi \in H^1(\mathbb{R}^n)$ and $\phi \in L^2(\mathbb{R}^n)$ be fixed, with $\operatorname{supp}\psi, \operatorname{supp}\phi \subset B_1$, $\|\nabla\psi\|_2 = \alpha$. For the sake of simplicity we normalize $\|\phi\|_{H^{-1}} = 1$.

Consider a transformation of ϕ and ψ of the type

$$
\text{(59)} \quad u_0(x) = \mu^{1-n/2}\psi(x/\mu), \quad \text{and} \quad v_0(x) = \beta\phi(x/\mu),
$$

where $\beta > 0$ and $\mu \geq 1$ will be chosen in the sequel. This transformation preserves $\|\nabla\psi\|_2$ and transforms the energy into

$$
\text{(60)} \quad E(0) = \frac{1}{2}\|\phi\|_2^2\beta^2\mu^n + \frac{1}{2}\alpha^2 - \frac{1}{p+1}\mu^{[2n-(n-2)(p+1)]/2}\|\psi\|_{p+1}^{p+1}.
$$

Since we need data with prescribed energy λ, the two parameters μ and β are related by

$$
\text{(61)} \quad \frac{1}{2}\|\phi\|_2^2\beta^2\mu^n + \frac{1}{2}\alpha^2 - \frac{1}{p+1}\mu^{[2n-(n-2)(p+1)]/2}\|\psi\|_{p+1}^{p+1} = \lambda.
$$

For μ sufficiently large, namely for $\mu \geq \bar{\mu}(p, n, \alpha, \lambda, \|\psi\|_{p+1})$, the above equation (61) has a (unique) positive solution $\beta = \beta(\mu)$. This argument, together with the transformation (59), reduces the choice of the data to a convenient choice for μ, such that the inequality (58) holds for a suitably chosen $t = t_1 \leq \min\{L, T_1\}$.

Let us note that by the transformation (59) we have

$$
\text{(62)} \quad \operatorname{supp} u_0 \subset B_\mu, \quad \operatorname{supp} v_0 \subset B_1 \subset B_\mu, \quad \text{so that } L = \mu,.
$$

As a consequence of (61) we have

$$
\text{(63)} \quad \beta \sim K\mu^{-(n-2)(p+1)/4} \quad \text{as } \mu \to \infty,
$$

where $K = K(\psi, \phi, p, \alpha, \lambda) > 0$. This shows that $\beta\mu^{n/2} \to \infty$ as $\mu \to \infty$ since $p < n/(n-2)$. Then

$$
\text{(64)} \quad \widetilde{E}(0) = \frac{1}{2}\beta^2\mu^n\|\phi\|_2^2 + \frac{1}{2}\alpha^2 \to \infty \quad \text{as } \mu \to \infty.
$$

Now we can point out one of the difficulties in the proof, which arises in the case $m > 1$. First $L = \mu$ must be large enough in order that equation (61) has a solution β, while on the other hand a large value L implies that the factor $L^{-n(m-1)/2}$ is small in (58). Moreover, the identity (52) and the limit (64) imply that $T_1 \to 0$ as $\mu \to \infty$, together with any $t \leq \max\{\mu, T_1\}$ for which (58) holds. When $m > 1$, the factor $CL^{-n(m-1)/2}t$ in (58), goes to zero as $\mu \to \infty$, at least as fast as $\mu^{-n(m-1)/2}T_1$, regardless of the value $t \leq \{\mu, T_1\}$. This forces the remainder term in (58) (namely the term inside the brackets) to compensate for this fast decrease. Namely, the main reason for the technical assumption $(p, m) \in D_n$ is the competition between the term $L^{-n(m-1)/2}t$ and the remainder term in (58).

In case $m = 1$ the term $L^{-n(m-1)/2}t$ reduces to t, so that the above mentioned difficulty does not arise. Simple calculations show that

$$(65) \qquad \|v_0\|_{H^{-1}} \sim C\beta\mu^{n/2} \to \infty \quad \text{as } \mu \to \infty.$$

Now, due to (14) and (61) we have $E(0) - E_{\text{cr}} = \lambda - E_{\text{cr}} \leq \lambda + 1$. Consequently since $t \leq T_1$ we have

$$(66) \qquad Ct(E(0) - E_{\text{cr}})^m \leq CT_1(\lambda + 1)^m$$

and then, by (65) and the fact that $T_1 \to 0$ as $\mu \to \infty$ we get

$$Ct(E(0) - E_{\text{cr}})^m \leq \frac{1}{2}\|v_0\|_{H^{-1}}^{m+1}$$

for μ large enough. This gives us a possibility to neglect the last term in the first inequality of (58) and to rewrite (58) as follows

$$(67) \qquad \int_0^t \|u_t\|_{m+1}^{m+1} \geq C\mu^{-n(m-1)/2}t\left[\|v_0\|_{H^{-1}}^{m+1} - Ct^{m+1}(\widetilde{E}(0) - E_{\text{cr}})^{p(m+1)/(p+1)}\right]$$
$$\geq E(0) - E_{\text{cr}}$$

for some $t \leq \min\{T_1, \mu\}$, and for $C = C(p, m, n, \alpha, \lambda, \phi, \psi) > 0$.

For the sake of positivity of the right hand side of the first inequality in (67), we choose t_1 in the form

$$(68) \qquad t_1 = \|v_0\|_{H^{-1}}\left(\widetilde{E}(0) - E_{\text{cr}}\right)^{-\nu}$$

where $\nu > 0$ has to be determined in the sequel. Since the estimate (67) is satisfied for $t \leq \min\{T_1, \mu\}$, we require that $t_1 \leq T_1$ for μ sufficiently large. Using the estimate

$$\|v_0\|_{H^{-1}} \leq \|v_0\|_2 \leq \widetilde{E}^{1/2}(0),$$

together with (52), (64) and (68) we get

$$(69) \qquad \nu > \gamma_0 - \frac{1}{2}.$$

For μ large enough we have $T_1 \leq \mu$ (see (52) and (64)) so the value of t_1 from (68) satisfies our requirements. We put $t = t_1$ in (67) and get

$$(70) \qquad \int_0^{t_1} \|u_t\|_{m+1}^{m+1} \geq C\mu^{-n(m-1)/2}\|v_0\|_{H^{-1}}^{m+2}\left(\widetilde{E}(0) - E_{\text{cr}}\right)^{-\nu}$$
$$\times \left[1 - C(\widetilde{E}(0) - E_{\text{cr}})^{(m+1)[p/(p+1)-\nu]}\right].$$

From the asymptotic behavior of $\widetilde{E}(0)$ as $\mu \to \infty$, and for the sake of positivity of the right hand side of (70), we require that

$$(71) \qquad 1 - C(\widetilde{E}(0) - E_{\text{cr}})^{(m+1)(p/(p+1)-\nu)} \geq C > 0.$$

The last inequality holds provided that

$$(72) \qquad \nu > \frac{p}{p+1}.$$

Then, for ν satisfying (69) and (72), we can simplify (67) by using (70) together with (71) in the following way

$$(73) \qquad \int_0^{t_1} \|u_t\|_{m+1}^{m+1} \geq C\mu^{-n(m-1)/2}\|v_0\|_{H^{-1}}^{m+2}\left(\widetilde{E}(0) - E_{\text{cr}}\right)^{-\nu} \geq E(0) - E_{\text{cr}}$$

for μ large enough. Taking into account the asymptotic behavior of $\|v_0\|_{H^{-1}}$ and $\widetilde{E}(0)$ together with (63) as $\mu \to \infty$, the estimate (73) holds for μ large enough provided that

$$(74) \qquad -\frac{n(m-1)}{2} + \frac{2n-(n-2)(p+1)}{4}(m+2) - \frac{2n-(n-2)(p+1)}{2}\nu > 0.$$

Solving the system (69), (72) and (74) leads to the restriction on m and p and the space dimension n which is expressed by the condition $(p,m) \in D_n$ (see (7)).

The existence of a solution μ of the elementary inequalities (69), (72) and (74) bring us to a contradiction with the assumption that the solution is global. This, together with the continuation principle (see [17]) implies finite time blow–up of the solution. Since this is true for μ large enough there is a continuous branch $\mu \to (u_0(\mu), v_0(\mu))$ of data satisfying (10) and (11) for which the solution of (1) blows up. $\qquad\square$

Proof of Corollary 1. We choose $\psi \in H^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, from the transformation (59) we get

$$(75) \qquad\qquad\qquad \|u_0\|_\infty = \mu^{1-n/2}\|\psi\|_\infty.$$

The argument used in the proof of Theorem 2 concludes the proof of the Corollary.
$\qquad\square$

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