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WEIGHTED L^2-ESTIMATES FOR DISSIPATIVE WAVE EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. We establish weighted L^2-estimates for the wave equation with variable damping $u_{tt} - \Delta u + au_t = 0$ in $\mathbf{R}^n$, where $a(x) \geq a_0(1+|x|)^{-\alpha}$ with $a_0 > 0$ and $\alpha \in [0, 1)$. In particular, we show that the energy of solutions decays at a polynomial rate $t^{-(n-\alpha)/(2-\alpha)-1}$ if $a(x) \sim a_0|x|^{-\alpha}$ for large $|x|$. We derive these results by strengthening significantly the multiplier method. This approach can be adapted to other hyperbolic equations with damping.

1. Introduction

The long time behavior of dissipative wave equations has been studied extensively as an important problem related to global well-posedness, existence of attractors and exact controllability. In contrast to the enormous number of works concerning L^2-estimates, the number of general techniques is quite small: (1) resolvent estimates for parameter dependent elliptic operators [AL], [CZ], [FZ], [HM], [RT1], [RW], [SJ], [W1], (2) observability inequalities combined with the energy dissipation law and semigroup property [BN], [DC], [DG], [DLG], [FJ], [HA], [LT], [KV], [MP], [MN], [NM], [RT2], [Z1], [Z2], or (3) first-order multipliers determined by the symmetries of wave equation and consistent with the boundary conditions [IM1], [IM2], [M2], [MN1], [MN2], [UH]. Since a complete survey of results exceeds the scope of this paper, many significant contributions are inevitably omitted.

Despite their differences, the existing methods have a common characteristic: they are developed to study solutions with exponential asymptotic behavior. Thus estimates for solutions with polynomial behavior are rarely precise. A typical case is the wave equation with damping $u_{tt} - \Delta u + au_t = 0$, where $a(x) \geq a_0 > 0$ in $\mathbf{R}^n$. When $a(x) = a_0$, Matsumura [M1] has established $\|u(t)\|_{L^2} = O(t^{-n/4})$ as $t \to \infty$. This estimate is obtained by simple Fourier analysis but its generalization to variable coefficients is far from straightforward. In fact, the necessary resolvent estimates within the first method are not found yet. The second method can not be applied either, since the observability inequality fails for $a \in L^\infty(\mathbf{R}^n)$, [DG]. Finally, the results of third method are usually dimension independent: $\|u(t)\|_{L^2} = O(1)$. Sharper estimates require more complicated arguments, such as [IM2] showing that $\|u(t)\|_{L^2} = O(t^{-1/2})$ for $n \geq 2$. This decay rate is the best possible for $n = 2$, but it is unlikely to be sharp for $n \geq 3$. The weakness of multiplier method may be attributed to the use of $tu_t + x \cdot \nabla x$ and other Morawetz-type expressions [MC] related to the scaling invariance of wave equation $u_{tt} - \Delta u = 0$. Examples like

[1]Research of the first author supported in part by NSF grant DMS-0245578
Date: October 19, 2007.
1991 *Mathematics Subject Classification.* Primary 35L05, 35L70; Secondary 37L15.
Key words and phrases. wave equation, weak dissipation, decay rates.

$a(x) = a_0$ show that the diffusion equation $a_0 u_t - \Delta u = 0$ is a much more accurate approximation; the "diffusion phenomenon" is studied in [MMR], [MY1], [MY2], [NT1], [NK1], and the references therein.

The goal of this paper is to strengthen the multiplier method for the wave equation with variable damping in $\mathbf{R}^n$. We obtain more precise L^2-estimates and faster decay rates depending on n. The multipliers are derived from positive approximate solutions in accordance with the diffusion phenomenon. Our computations are elementary and sufficiently flexible to allow other second-order hyperbolic equations with damping. The new estimates help investigate the existence and behavior of global solutions for various nonlinear perturbations, [TY].

Given a positive coefficient $a \in C^1(\mathbf{R}^n)$, such that

$$a(x) \geq a_0(1+|x|)^{-\alpha}, \quad \alpha \in [0,1), \tag{1.1}$$

we study the Cauchy problem

$$\begin{aligned} &u_{tt} - \Delta u + a u_t = 0, \quad x \in \mathbf{R}^n, \quad t > 0, \\ &u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \end{aligned} \tag{1.2}$$

For simplicity the initial data are compactly supported in the energy space:

$$u_0 \in H^1(\mathbf{R}^n), \quad u_1 \in L^2(\mathbf{R}^n), \quad u_0(x) \text{ and } u_1(x) = 0 \text{ for } |x| > R.$$

It is well known that (1.2) admits a unique weak solution u satisfying

$$u \in C((0,\infty), H^1(\mathbf{R}^n)), \quad u_t \in C((0,\infty), L^2(\mathbf{R}^n)),$$

and having compact support,

$$u(t,x) = 0 \ \text{ for } \ |x| > t + R.$$

The main quantities of interest are the L^2-norm and energy associated with u. In fact, the energy arises after multiplying equation (1.2) with u_t and applying the divergence theorem on $[0,t] \times \mathbf{R}^n$:

$$\int (u_t^2 + |\nabla u|^2)\, dx = \int (u_t^2 + |\nabla u|^2)\, dx \Big|_{t=0} - 2\int_0^t \int a u_s^2 \, dx ds.$$

Hence the energy is a non-increasing function of t. The important question is whether the energy decays as $t \to \infty$ and if so, how fast it decays. Affirmative answers require some lower bound on $a(x)$ as $|x| \to \infty$, since solutions are asymptotically free if $a(x) \leq a_0(1+|x|)^{-1-\delta}$ with $\delta > 0$; see Mochizuki [MK]. Of course the energy of such scattering solutions will approach a non-zero constant as $t \to \infty$. Moreover, the energy will not decay uniformly with respect to initial data (u_0, u_1) if $a(x)$ vanishes in a neighborhood of a line ω, as shown by Kawashita, Nakazawa, and Soga [KNS].

A simple idea how to strengthen the multiplier method for equation (1.2) comes from the special cases $a = a_0$ and $a = a(t)$ where the diffusion phenomenon is well understood. In the former case, $u(t,x) \sim C_0 w(t,x)$ with C_0 depending on the initial data and w being a solution of the diffusion equation

$$w_t - \Delta w = 0, \quad x \in \mathbf{R}^n, \quad t > 0.$$

More details are given in [NT1]. It turns out that we can use the Gaussian

$$w(t,x) = t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

A similar asymptotics holds in the latter case of (1.2) studied by Wirth [W3]. Here the precise conditions for $u(t,x) \sim C_0 w(t,x)$ and exact constant C_0 are more subtle but the important fact is that w solves

$$a(t)w_t - \Delta w = 0, \quad x \in \mathbf{R}^n, \quad t > 0.$$

Thus $w^{-1}u \sim C_0$ on the support of u in both cases. We expect the diffusion phenomenon to persist when $a = a(x)$ in (1.2), so we will try to obtain sharper L^2-estimates by factoring out the asymptotic profile w and working with $w^{-1}u$. This quotient will admit more precise estimates since it will vary relatively slowly. A slight difficulty is the lack of explicit solutions to

$$a(x)w_t - \Delta w = 0, \quad x \in \mathbf{R}^n, \quad t > 0,$$

but we can choose w satisfying the corresponding inequality with a greater left side. The new problem has simple solutions of the form

$$w(t,x) = t^{-m_1} e^{-m_2 \frac{A(x)}{t}}, \tag{1.3}$$

with parameters m_1, m_2, and a function $A(x)$ given by (1.5) below. Unexpectedly, we can still estimate $w^{-1}u$ very precisely. The only multiplier has two terms:

$$w_1(w^{-1}u)_t + w_0(w^{-1}u), \tag{1.4}$$

where the weights are defined by

$$w_0 = w \quad \text{and} \quad w_1 = m_3 w \left(\frac{m_4}{t} + \frac{w_t}{w} \right)^{-1},$$

for certain constants m_3, m_4, and an approximate solution $w(t,x)$ from (1.3). Here the role of m_4 is to make the denominator of w_1 positive, while the role of w_1 itself is to equalize the two terms in (1.4). Let us also mention that m_1 in (1.3) determines the decay rates of weighted energy and L^2-norms.

To state the results, we begin with conditions on a. First, we require (1.1). We also assume that there exists a solution of the Poisson equation

$$\Delta A(x) = a(x), \quad x \in \mathbf{R}^n, \tag{1.5}$$

with the following properties:

$$\begin{aligned}
&(a1) \qquad A(x) \geq 0 \ \text{ for all } \ x, \\
&(a2) \qquad A(x) = O(|x|^{2-\alpha}) \ \text{ for large } \ |x|, \\
&(a3) \qquad m(a) = \liminf_{x \to \infty} \frac{a(x)A(x)}{|\nabla A(x)|^2} \ > \ 0.
\end{aligned}$$

Such solutions $A(x)$ exist in many cases, including radial coefficients $a(x)$ which behave like $|x|^{-\alpha}$ as $|x| \to \infty$. We consider examples in Proposition 1.3. For non-radial coefficient $a(x)$ the solution $A(x)$ of the Poisson equation (1.5) can be constructed by the method in [WZ].

The main decay estimates of this paper are stated next.

Theorem 1.1. *Assume that conditions (1.1) and (a1)–(a3) hold. Then for every $\delta > 0$ the solution of (1.2) satisfies*

$$\begin{aligned}
\int e^{(m(a)-\delta)\frac{A(x)}{t}} a(x) u^2 \, dx \quad &\leq \quad C_\delta (\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2) t^{\delta - m(a)}, \\
\int e^{(m(a)-\delta)\frac{A(x)}{t}} (u_t^2 + |\nabla u|^2) \, dx \quad &\leq \quad C_\delta (\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2) t^{\delta - m(a) - 1},
\end{aligned}$$

for all $t \geq 1$. The constant C_δ depends also on R, a, and n.

Hence $m(a)$ from definition $(a3)$ determines the decay rates. Moreover, $m(a)$ is invariant under scaling and depends only on the behavior of $a(x)$ as $|x| \to \infty$. Additional assumptions on a will probably allow $\delta = 0$ in Theorem 1.1.

Another important consequence of the main theorem is that all norms under consideration, restricted to $\{x : A(x) \geq t^{1+\epsilon}\}$ with $\epsilon > 0$, decay exponentially. Notice that these regions overlap with the support $\{x : |x| \leq t + R\}$.

Corollary 1.2. *Assume that conditions (1.1) and (a1)–(a3) hold. Then for every $\delta > 0$ and $\epsilon > 0$ the solution of (1.2) satisfies*

$$\int_{A(x)\geq t^{1+\epsilon}} (u^2 + u_t^2 + |\nabla u|^2)\, dx \quad \leq \quad C_\delta(\|\nabla u_0\|^2_{L^2} + \|u_1\|^2_{L^2}) e^{-(m(a)-\delta)t^\epsilon},$$

where $t \geq 1$.

The most interesting applications of Theorem 1.1 rely on two-sided inequalities or asymptotics for $a(x)$. Such examples yield more information for $A(x)$. The following result is shown in Appendix B.

Proposition 1.3. *Let a be a radially symmetric function in $C^1(\mathbf{R}^n)$, $n \geq 1$, which satisfies*

$$a_0(1+|x|)^{-\alpha} \leq a(x) \leq a_1(1+|x|)^{-\alpha}, \quad \alpha \in [0,1). \tag{1.6}$$

(i) Equation (1.5) admits a solution $A \in C^3(\mathbf{R}^n)$, such that

$$(A1) \qquad A_0(1+|x|)^{2-\alpha} \leq A(x) \leq A_1(1+|x|)^{2-\alpha},$$

$$(A2) \qquad m(a) > 0,$$

where A_0 and A_1 are positive constants.

(ii) In the special case

$$a(x) \sim a_2|x|^{-\alpha}, \quad |x| \to \infty, \tag{1.7}$$

with $a_2 > 0$, equation (1.5) has a solution with the following properties:

$$(A3) \qquad A(x) \sim \frac{a_2}{(2-\alpha)(n-\alpha)}|x|^{2-\alpha}, \quad |x| \to \infty,$$

$$(A4) \qquad m(a) = \frac{n-\alpha}{2-\alpha}.$$

Combining the above proposition with Theorem 1.1, we can give more explicit weighted estimates.

Corollary 1.4. *Assume that a is a radial C^1-function satisfying condition (1.6). Then for every $\delta > 0$ the solution of (1.2) satisfies*

$$\int e^{A_0(m(a)-\delta)\frac{|x|^{2-\alpha}}{t}} u^2\, dx \quad \leq \quad C_\delta(\|\nabla u_0\|^2_{L^2} + \|u_1\|^2_{L^2}) t^{\delta + \frac{\alpha}{2-\alpha} - m(a)},$$

$$\int e^{A_0(m(a)-\delta)\frac{|x|^{2-\alpha}}{t}} (u_t^2 + |\nabla u|^2)\, dx \quad \leq \quad C_\delta(\|\nabla u_0\|^2_{L^2} + \|u_1\|^2_{L^2}) t^{\delta - m(a) - 1},$$

where $t \geq 1$.

Corollary 1.5. *Assume that a is a radial C^1-function satisfying conditions (1.6) and (1.7). Then for every $\delta > 0$ the solution of (1.2) satisfies*

$$\begin{aligned}\int e^{a_2(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}} u^2\, dx &\le C_\delta(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2)t^{\delta-\frac{n-2\alpha}{2-\alpha}},\\ \int e^{a_2(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}} (u_t^2 + |\nabla u|^2)\, dx &\le C_\delta(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2)t^{\delta-\frac{n-\alpha}{2-\alpha}-1},\end{aligned}$$

where $t \ge 1$.

Let us mention that the energy and L^2-norm in Corollary 1.5 have faster decay rates when $\alpha \to 1$. Thus, weaker damping coefficients give rise to solutions which decay faster at infinity. This seems counterintuitive, but it only means that the constant damping $\alpha = 0$ is an "overdamping" in equation (1.2).

The results of this paper should be put into perspective with previous results on several related problems. One of them is the asymptotic behavior of energy and other norms when $a = a(t)$ in the main equation (1.2). Using Fourier analysis, Wirth [W2], [W3], Reissig [MR], and Reissig and Wirth [RW] have obtained a series of sharp $L^p - L^q$ estimates including

$$\begin{aligned}\int u^2\, dx &\le C_0 t^{-(1-k)\frac{n}{2}},\\ \int (u_t^2 + |\nabla u|^2)\, dx &\le C_0 t^{-(1-k)(\frac{n}{2}+1)},\end{aligned}$$

for $a(t) = a_0(1+t)^k$ and $k \in (-1, 1)$. Notice how the decay increases although the damping coefficient weakens as $k \to -1$. The absence of x in $a(t)$ allows a partial Fourier transform and a WKB-representation of u by Fourier multipliers. This approach can generalize not only Matsumura's estimates [M1] but also the more precise estimates in [NT2], [NK2] for $a = 1$ and Strichartz's estimates [SR] for $a = 0$.

Another interesting problem is to treat coefficients with critical decay in (1.2). The main condition on $a = a(t, x)$ is

$$a(t, x) \ge a_0(1 + t + |x|)^{-1},$$

where $a_0 > 0$. Here Fourier analysis becomes cumbersome as it involves localizations in both frequency space and extended phase space $(0, \infty) \times \mathbf{R}^n$. The multiplier method is a natural alternative applied by Matsumura [M2] and Uesaka [UH] to establish the decay estimate

$$\int (u_t^2 + |\nabla u|^2)\, dx \le C_0 t^{-\min\{2, a_0\}}. \tag{1.8}$$

Their multipliers have the form $\{w(t)u\}_t$ with a suitable weight $w(t)$. Remarkably this method can handle the critical decay but can not take advantage of any slower decay, such as (1.1). In fact estimate (1.8) does not remain sharp for stronger dissipations; for instance Corollary 1.5 shows that the decay rate is close to $t^{-n/2-1}$ if α is close to 0.

The final question is about the behavior of exterior energy defined by restricting the solution $u(t, x)$ to large $|x|$ depending on t. Ikehata [IR] has considered coefficients

$$a(x) = \frac{a_0}{|x|^\alpha}, \quad \alpha \in [0, 1),$$

and established the following weighted estimate for (1.2):

$$\int e^{2a_0(2-\alpha)^{-2}\frac{|x|^{2-\alpha}}{t}}(u_t^2+|\nabla u|^2)\,dx \le C_0,$$

where $t \ge 1$. Hence the solution decays very fast if $|x|^{2-\alpha}/t$ is large. If this ratio is small, however, the result provides little new information.

The paper is organized as follows. In Section 2 we state a general weighted identity whose proof is postponed to Appendix A. We derive a modified equation for $w^{-1}u$ and find sufficient conditions for its energy to decrease in Section 3. The existence of non-trivial weights is shown in Section 4. The last Section 5 contains short proofs of Theorem 1.1 and Corollaries 1.2, 1.4, and 1.5. For completeness we verify Proposition 1.3 in Appendix B.

2. A Weighted Energy Identity

Let us consider a general first-order perturbation of the wave equation:

$$u_{tt}-\Delta u+au_t+b\cdot\nabla u+cu=0, \tag{2.1}$$

where the coefficients a, $b=(b_1,b_2,\dots,b_n)$, and c are C^1-functions. The above form is invariant under multiplicative perturbations of u. Our goal is to derive a weighted identity for u involving three positive C^2-functions: w_0, w_1 and w.

Define $\hat{u}=w^{-1}u$, or $u=w\hat{u}$. Substituting this product into (2.1), we have

$$\hat{u}_{tt}-\Delta\hat{u}+\hat{a}\hat{u}_t+\hat{b}\cdot\nabla\hat{u}+\hat{c}\hat{u}=0$$

with new coefficients

$$\begin{aligned}&\hat{a}=a+2w^{-1}w_t, \quad \hat{b}=b-2w^{-1}\nabla w,\\ &\hat{c}=w^{-1}(w_{tt}-\Delta w+aw_t+b\cdot\nabla w+cw).\end{aligned} \tag{2.2}$$

We will multiply the transformed equation with $w_1\hat{u}_t+w_0\hat{u}$, rearrange terms, and integrate by parts over $\mathbf{R}^n$. The resulting identity is shown below.

Proposition 2.1. *Let u be a solution of (2.1) with compactly supported data*

$$u_0\in H^2, \quad u_1\in H^1.$$

Assume that w_0, w_1, and $w>0$ are C^2-functions. Then

$$\frac{d}{dt}E(\hat{u}_t,\nabla\hat{u},\hat{u})+F(\hat{u}_t,\nabla\hat{u})+G(\hat{u})=0,$$

where

$$
\begin{aligned}
E(\hat{u}_t, \nabla\hat{u}, \hat{u}) &= \frac{1}{2}\int [w_1(\hat{u}_t^2 + |\nabla\hat{u}|^2) + 2w_0\hat{u}_t\hat{u} + (\hat{c}w_1 - \partial_t w_0 + \hat{a}w_0)\hat{u}^2]\ dx,\\
F(\hat{u}_t, \nabla\hat{u}) &= \frac{1}{2}\int (-\partial_t w_1 + 2\hat{a}w_1 - 2w_0)\hat{u}_t^2\ dx\\
&\quad + \int (\nabla w_1 + \hat{b}w_1)\cdot \hat{u}_t\nabla\hat{u}\ dx\\
&\quad + \frac{1}{2}\int (-\partial_t w_1 + 2w_0)|\nabla\hat{u}|^2\ dx,\\
G(\hat{u}) &= \frac{1}{2}\int [\partial_t^2 w_0 - \Delta w_0 - \partial_t(\hat{a}w_0) - \nabla\cdot(\hat{b}w_0) + 2\hat{c}w_0 - \partial_t(\hat{c}w_1)]\hat{u}^2\ dx.
\end{aligned}
$$

The coefficients $\hat{a}$, $\hat{b}$, and $\hat{c}$ are given in (2.2).

Proof. This involves elementary but tedious calculations shown in Appendix A.

3. Wave Equations with Weak Dissipation

We can apply the identity in Proposition 2.1 to the main equation (1.2). Then

$$a = a(x), \quad b = 0, \quad c = 0,$$

so the transformed coefficients (2.2) become

$$
\begin{aligned}
&\hat{a} = a + 2w^{-1}w_t, \quad \hat{b} = -2w^{-1}\nabla w,\\
&\hat{c} = w^{-1}(w_{tt} - \Delta w + aw_t).
\end{aligned}
\tag{3.1}
$$

The weighted identity for $\hat{u}$ in Proposition 2.1 simplifies if we choose $w_0 = w$. There is no simple expression for w_1 in terms of w, so w_1 will be chosen later. It is convenient to keep the most complex coefficient $\hat{c}$ and express the other coefficients in terms of w, w_1, and a. The simplified functionals are

$$
\begin{aligned}
E(\hat{u}_t, \nabla\hat{u}, \hat{u}) &= \frac{1}{2}\int [w_1(\hat{u}_t^2 + |\nabla\hat{u}|^2) + 2w\hat{u}_t\hat{u} + (\hat{c}w_1 + \partial_t w + aw)\hat{u}^2]\ dx,\\
F(\hat{u}_t, \nabla\hat{u}) &= \frac{1}{2}\int (-\partial_t w_1 + 2aw_1 + 4w_1\partial_t \ln w - 2w)\hat{u}_t^2\ dx\\
&\quad + \int (\nabla w_1 - 2w_1\nabla \ln w)\cdot \hat{u}_t\nabla\hat{u}\ dx\\
&\quad + \frac{1}{2}\int (-\partial_t w_1 + 2w)|\nabla\hat{u}|^2\ dx,\\
G(\hat{u}) &= \frac{1}{2}\int [\hat{c}w - \partial_t(\hat{c}w_1)]\hat{u}^2\ dx.
\end{aligned}
$$

These satisfy the identity

$$\frac{d}{dt}E(\hat{u}_t, \nabla\hat{u}, \hat{u}) + F(\hat{u}_t, \nabla\hat{u}) + G(\hat{u}) = 0, \tag{3.2}$$

for initial data $u_0 \in H^2$ and $u_1 \in H^1$. However, we will establish an integral version of this inequality for all data in the energy space. Meanwhile we need more conditions on the damping and weights to insure $F \geq 0$ and $G \geq 0$, so $dE/dt \leq 0$.

Proposition 3.1. *Let w and w_1 be positive weights and $\hat{c}$ be defined in (3.1). Assume that*

(i) $\quad \hat{c} \geq 0, \quad \partial_t \hat{c} \leq 0,$

(ii) $\quad -\partial_t w_1 + w \geq 0,$

(iii) $\quad (-\partial_t w_1 + 2w)(-\partial_t w_1 + 2aw_1 + 4w_1 \partial_t \ln w - 2w) \geq (\nabla w_1 - 2w_1 \nabla \ln w)^2.$

The weighted energy $E(\hat{u}_t, \nabla \hat{u}, \hat{u})$ is a non-increasing function of t for every solution u of (1.2):

$$\frac{1}{2} \int [w_1(\hat{u}_t^2 + |\nabla \hat{u}|^2) + 2w\hat{u}_t\hat{u} + (\hat{c}w_1 + \partial_t w + aw)\hat{u}^2] \, dx \leq E_0$$

for all $t \geq t_0$, where E_0 is defined as $E_0 = E(\hat{u}_t, \nabla \hat{u}, \hat{u})|_{t=t_0}$. (Recall that $\hat{u} = w^{-1}u$.)

Proof. If the data are compactly supported and satisfy $u_0 \in H^2$ and $u_1 \in H^1$, identity (3.2) holds. Notice that conditions (i) and (ii) imply

$$\hat{c}w - \partial_t(\hat{c}w_1) = \hat{c}(w - \partial_t w_1) - (\partial_t \hat{c})w_1 \geq 0.$$

Hence $G(\hat{u}) \geq 0$. Condition (iii) and $-\partial_t w_1 + 2w \geq 0$, which follows from (ii), guarantee that the quadratic form $F(\hat{u}_t, \nabla \hat{u}) \geq 0$. Thus (3.2) yields $dE(\hat{u}_t, \nabla \hat{u}, \hat{u})/dt \leq 0$ or $E(\hat{u}_t, \nabla \hat{u}, \hat{u}) \leq E_0$.

In general, any compactly supported data $u_0 \in H^1$ and $u_1 \in L^2$ can be approximated by compactly supported C^∞ sequences $u_0^{(k)} \to u_0$ in H^1 and $u_1^{(k)} \to u_1$ in L^2. Denote the corresponding solutions of (1.2) by $u^{(k)}$ and their weighted energy by $E(\hat{u}_t^{(k)}, \nabla \hat{u}^{(k)}, \hat{u}^{(k)})$. The first part of this proof shows that

$$E(\hat{u}_t^{(k)}, \nabla \hat{u}^{(k)}, \hat{u}^{(k)}) \leq E(\hat{u}_t^{(k)}, \nabla \hat{u}^{(k)}, \hat{u}^{(k)})|_{t=t_0}. \tag{3.3}$$

Since the weights w and w_1 are bounded positive functions on every finite interval of t, the weighted energy is equivalent to the standard energy:

$$\begin{aligned} & |E^{1/2}(\hat{u}_t^{(k)}, \nabla \hat{u}^{(k)}, \hat{u}^{(k)}) - E^{1/2}(\hat{u}_t, \nabla \hat{u}, \hat{u})| \\ \leq \quad & C(T) \left(\|u_t^{(k)} - u_t\|_{L^2} + \|\nabla u^{(k)} - \nabla u\|_{L^2} \right), \end{aligned}$$

whenever $t \in [0, T]$. It is well known that the weak solutions of problem (1.2) satisfy

$$\|u_t^{(k)} - u_t\|_{L^2} + \|\nabla u^{(k)} - \nabla u\|_{L^2} \quad \leq \quad C \left(\|u_1^{(k)} - u_1\|_{L^2} + \|\nabla u_0^{(k)} - \nabla u_0\|_{L^2} \right).$$

Hence we obtain

$$\begin{aligned} & |E^{1/2}(\hat{u}_t^{(k)}, \nabla \hat{u}^{(k)}, \hat{u}^{(k)}) - E^{1/2}(\hat{u}_t, \nabla \hat{u}, \hat{u})| \\ \leq \quad & C(T) \left(\|u_1^{(k)} - u_1\|_{L^2} + \|\nabla u_0^{(k)} - \nabla u_0\|_{L^2} \right). \end{aligned}$$

We can use the latter estimate to pass to the limit as $k \to \infty$ in (3.3):

$$E(\hat{u}_t, \nabla \hat{u}, \hat{u}) = \lim_{k \to \infty} E(\hat{u}_t^{(k)}, \nabla \hat{u}^{(k)}, \hat{u}^{(k)}) \leq \lim_{k \to \infty} E(\hat{u}_t^{(k)}, \nabla \hat{u}^{(k)}, \hat{u}^{(k)})|_{t=t_0} = E_0.$$

This completes the proof for general data in the energy space.

It is essential to have a positive definite weighted energy $E(\hat{u}_t, \nabla\hat{u}, \hat{u})$. The next step is to find some sufficient conditions. We will rely on an auxiliary estimate derived from the inequality $E(\hat{u}_t, \nabla\hat{u}, \hat{u}) \le E_0$ in Proposition 3.1. In fact, we can neglect a few non-negative terms and combine $2w\partial_t\hat{u}\hat{u} + \partial_t w\hat{u}^2 = \partial_t(w\hat{u}^2)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\int w\hat{u}^2\,dx + \frac{1}{2}\int aw\hat{u}^2\,dx \le E_0. \tag{3.4}$$

This is sufficient for an upper bound on the weighted L^2-norm if a satisfies (1.1).

Proposition 3.2. *Let w and w_1 be positive weights satisfying conditions (i)–(iii) in Proposition 3.1. Assume that a satisfies (1.1). Then the following estimate holds for the solution u of problem (1.2):*

$$\int w\hat{u}^2\,dx \le N_0 + CE_0t^\alpha$$

for $t \ge t_0$, where $N_0 = \int w\hat{u}^2\,dx\big|_{t=t_0}$.

Proof. Since $|x| \le t + R$ on the support of $\hat{u}$, (3.4) and (1.1) yield

$$\frac{d}{dt}\int w\hat{u}^2\,dx + \frac{a_0}{(1+t+R)^\alpha}\int w\hat{u}^2\,dx \le 2E_0.$$

We rewrite the inequality as

$$\frac{d}{dt}\left(e^{\frac{a_0}{1-\alpha}(1+t+R)^{1-\alpha}}\int w\hat{u}^2\,dx\right) \le 2E_0 e^{\frac{a_0}{1-\alpha}(1+t+R)^{1-\alpha}}$$

and solve it for the weighted L^2-norm using the estimate

$$\int_{t_0}^{t} e^{C\tau^{1-\alpha}}d\tau = \frac{1}{1-\alpha}\int_{t_0^{1-\alpha}}^{t^{1-\alpha}} e^{Cz}z^{\frac{\alpha}{1-\alpha}}dz \le \frac{t^\alpha e^{Ct^{1-\alpha}}}{C(1-\alpha)}.$$

We can now state the conditions for positive definiteness of $E(\hat{u}_t, \nabla\hat{u}, \hat{u})$. This result is a consequence of Propositions 3.1 and 3.2.

Proposition 3.3. *Assume that condition (1.1) on a holds. If w and w_1 satisfy conditions (i)–(iii) in Proposition 3.1 and the additional conditions*

$$\begin{aligned} &(iv) \qquad w \le C_1 t^{-\alpha} w_1, \\ &(v) \qquad \partial_t w \ge -C_1 t^{-\alpha} w, \end{aligned}$$

with $C_1 > 0$, then the solution u of (1.2) satisfies

$$\begin{aligned} \int w_1(\hat{u}_t^2 + |\nabla\hat{u}|^2)\,dx &\le C(N_0 + E_0), \\ \int aw\hat{u}^2\,dx &\le C(N_0 + E_0), \end{aligned}$$

for $t \ge t_0$. (Here E_0 and N_0 are defined in Propositions 3.1 and 3.2, respectively.)

Proof. We begin with the result of Proposition 3.1:

$$\int [w_1(\hat{u}_t^2 + |\nabla\hat{u}|^2) + 2w\hat{u}_t\hat{u} + (\hat{c}w_1 + \partial_t w + aw)\hat{u}^2]\,dx \le 2E_0.$$

Estimating $|2w\hat{u}_t\hat{u}| \leq \epsilon t^\alpha w\hat{u}_t^2 + \epsilon^{-1}t^{-\alpha}w\hat{u}^2$, where $\epsilon > 0$, we obtain

$$\int [(w_1 - \epsilon t^\alpha w)(\hat{u}_t^2 + |\nabla\hat{u}|^2) + (\hat{c}w_1 + \partial_t w + aw - \epsilon^{-1}t^{-\alpha}w)\hat{u}^2]\ dx \leq 2E_0.$$

This inequality and conditions (iv), (v) allow us to write

$$\begin{aligned} & (1-\epsilon C_1)\int [w_1(\hat{u}_t^2 + |\nabla\hat{u}|^2)\ dx + \int (\hat{c}w_1 + aw)\hat{u}^2\ dx \\ \leq\ & 2E_0 + (C_1 + \epsilon^{-1})t^{-\alpha}\int w\hat{u}^2\ dx. \end{aligned}$$

To complete the proof, it is sufficient to choose $\epsilon = (2C_1)^{-1}$ and estimate the right side by Proposition 3.2. Actually we derive a stronger estimate as $\hat{c}w_1\hat{u}^2 \geq 0$.

Proposition 3.3 shows that the L^2-norm of the original solution u satisfies

$$\int aw^{-1}u^2\ dx \leq C(N_0 + E_0), \tag{3.5}$$

since $\hat{u} = w^{-1}u$. The corresponding estimates of u_t and ∇u in terms of $\hat{u}_t$ and $\nabla\hat{u}$ are not immediate. In fact we make another assumption on w and w_1. The final estimates are given below.

Proposition 3.4. *Assume that conditions (1.1) and (i)–(v) hold. If*

$$(vi) \qquad w_1 w^{-3}(w_t^2 + |\nabla w|^2) \leq C_2 a(x)$$

holds for some $C_2 > 0$, the solution u of problem (1.2) satisfies

$$\begin{aligned} \int aw^{-1}u^2\ dx &\leq C(N_0 + E_0), \\ \int w_1 w^{-2}(u_t^2 + |\nabla u|^2)\ dx &\leq C(N_0 + E_0), \end{aligned}$$

for all $t \geq t_0$.

Proof. The first estimate is just (3.5). To prove the estimate of u_t and ∇u, we notice that

$$\hat{u}_t^2 = (-w^{-2}w_t u + w^{-1}u_t)^2 \geq \frac{1}{2}w^{-2}u_t^2 - 3w^{-4}w_t^2u^2$$

and

$$|\nabla\hat{u}|^2 = (-w^{-2}\nabla w u + w^{-1}\nabla u)^2 \geq \frac{1}{2}w^{-2}|\nabla u|^2 - 3w^{-4}|\nabla w|^2u^2.$$

Thus we have

$$\frac{1}{2}w_1w^{-2}(u_t^2 + |\nabla u|^2) \leq w_1(\hat{u}_t^2 + |\nabla\hat{u}|^2) + 3w_1w^{-4}(w_t^2 + |\nabla w|^2)u^2.$$

Integrating the inequality and applying the estimate in Proposition 3.3 yield

$$\frac{1}{2}\int w_1w^{-2}(u_t^2 + |\nabla u|^2)\ dx \leq C(N_0 + E_0) + 3\int w_1w^{-4}(w_t^2 + |\nabla w|^2)u^2\ dx.$$

Condition (vi) implies that the integral on the left side is bounded by

$$3C_2\int aw^{-1}u^2\ dx \leq C(N_0 + E_0),$$

as a consequence of estimate (3.5). The proof is complete.

Let us point out that the above result yields non-trivial estimates if the weights not only have properties $(i)-(vi)$ but also decay sufficiently fast as t and $|x| \to \infty$. For instance, the non-decaying weights $w = 1$ and $w_1 = t$ are admissible but the estimates in Proposition 3.4 are rather weak for this choice:

$$\begin{aligned} \int au^2\, dx &\leq C(N_0 + E_0), \\ \int (u_t^2 + |\nabla u|^2)\, dx &\leq C(N_0 + E_0)t^{-1}. \end{aligned}$$

The existence of more useful weights is related with the properties of solutions of the Poisson equation in $\mathbf{R}^n$. This problem is studied in the next section.

4. Construction of Weights

We will find non-trivial weights w and w_1 which meet conditions $(i)-(vi)$ in the previous section. Recall that w will be an approximate solution of equation (1.2). The second weight is a more complex function similar to $\int_1^t w(s,x)\, ds$.

It is crucial to have a solution of the Poisson equation

$$\Delta A(x) = a(x), \quad x \in \mathbf{R}^n,$$

with properties $(a1)-(a3)$ shown in the introduction. Given a small $\delta \in (0, \frac{1}{2}m(a))$ and a large $S_0 > 0$, we set

$$m = m(a) - 2\delta, \quad S(x) = (m(a) - \delta)A(x) + S_0. \tag{4.1}$$

Now the two weights are defined by

$$w(t,x) = t^{-m} e^{-\frac{S(x)}{t}}, \quad w_1(t,x) = \frac{3}{4}\left(\frac{6}{t} + \frac{S(x)}{t^2}\right)^{-1} w(t,x). \tag{4.2}$$

Constants such as S_0, $\frac{3}{4}$ and 6 are introduced for technical reasons. However, $\delta > 0$ plays an essential role in the proof of condition (i), i.e., $\hat{c} \geq 0$ and $\partial_t \hat{c} \leq 0$. This parameter leads to the loss of t^δ in our decay estimates.

A few useful properties of m and $S(x)$ are collected below.

Lemma 4.1. *Define m and $S(x)$ by (4.1) and assume that $m(a)$ and $A(x)$ have properties $(a1)-(a3)$. There exists $S_0 > 0$ such that*

$$\begin{array}{ll} (S1) & \Delta S(x) = (m+\delta)a(x) \ \textit{ for all } \ x, \\ (S2) & S(x) = O(|x|^{2-\alpha}) \ \textit{ for large } \ |x|, \\ (S3) & \left(1 - \dfrac{\delta}{2m(a)}\right) a(x)S(x) - |\nabla S(x)|^2 \geq 0 \ \textit{ for all } \ x. \end{array}$$

Proof. Property $(S1)$ follows from (4.1) and $\Delta A(x) = a(x)$. Property $(S2)$ is a restatement of $(a2)$. To verify $(S3)$, we fix $\epsilon > 0$ and use $(a3)$ to obtain the inequality

$$\frac{1+\epsilon}{m(a)} a(x)A(x) - |\nabla A(x)|^2 \geq 0$$

for sufficiently large $|x|$. Multiplying through with $(m(a) - \delta)^2$, we have

$$\frac{(1+\epsilon)(m(a)-\delta)}{m(a)} a(x)\tilde{S}(x) - |\nabla \tilde{S}(x)|^2 \geq 0$$

with $\tilde{S}(x) = (m(a)-\delta)A(x)$. We can choose $\epsilon = \delta/(2(m(a)-\delta))$. Inequality $(S3)$ holds with $S(x) = \tilde{S}(x)$ if $|x|$ is large. If we add a constant S_0 depending on δ, $S(x) = \tilde{S}(x) + S_0$ will satisfy $(S3)$ for all x.

Next we calculate the first and second order derivatives of w :

$$
\begin{aligned}
& w_t = \left(-\frac{m}{t} + \frac{S(x)}{t^2}\right) w, \quad w_{tt} = \left(-\frac{m}{t} + \frac{S(x)}{t^2}\right)^2 w + \left(\frac{m}{t^2} - \frac{2S(x)}{t^3}\right) w, \\
& \nabla w = -\frac{\nabla S(x)}{t}\, w, \quad \Delta w = \left(-\frac{\Delta S(x)}{t} + \frac{|\nabla S(x)|^2}{t^2}\right) w.
\end{aligned}
\tag{4.3}
$$

From these formulas we can express $\hat{c}$ in terms of m and S.

Lemma 4.2. *Let w and w_1 be defined by (4.2) and $\hat{c}$ be defined by (3.1). Then*

$$
\begin{aligned}
\hat{c} \quad &= \quad \frac{\Delta S - ma}{t} + \frac{aS - |\nabla S|^2}{t^2} \\
&\quad + \left(-\frac{m}{t} + \frac{S(x)}{t^2}\right)^2 + \frac{m}{t^2} - \frac{2S(x)}{t^3}.
\end{aligned}
$$

Proof. We substitute (4.3) into (3.1).

Finally we check a simple but useful upper bound on $\partial_t w_1/w_1$ to be used later.

Lemma 4.3. *Let w_1 be defined in (4.2). Then*

$$
\frac{\partial_t w_1(t,x)}{w_1(t,x)} \quad \le \quad \frac{-m+1}{t} + \frac{4}{3}\frac{S(x)}{t^2}.
$$

Proof. We calculate $\partial_t w_1/w_1 = \partial_t \ln w_1$ using definition (4.2):

$$
\frac{\partial_t w_1(t,x)}{w_1(t,x)} = -\frac{m}{t} + \frac{S(x)}{t^2} + \left(\frac{6}{t} + \frac{2S(x)}{t^2}\right)\left(6 + \frac{S(x)}{t}\right)^{-1}.
$$

Replacing $(6 + S(x)/t)^{-1}$ by $1/6$, we obtain

$$
\frac{\partial_t w_1(t,x)}{w_1(t,x)} \le -\frac{m}{t} + \frac{S(x)}{t^2} + \left(\frac{6}{t} + \frac{2S(x)}{t^2}\right)\cdot\frac{1}{6} = \frac{-m+1}{t} + \frac{4}{3}\frac{S(x)}{t^2}.
$$

We can now confirm that our choice of weights leads to the estimates in Section 3.

Proposition 4.4. *Assume that $A(x)$ satisfies $(a1)-(a3)$. Let w and w_1 be defined in (4.2) with m and $S(x)$ defined at (4.1). Then conditions (i)–(vi) in Section 3 hold for sufficiently large $t \ge t_0$.*

Proof. (i) The formula for $\hat{c}$ in Lemma 4.2 and the properties of $S(x)$ in Lemma 4.1 show that

$$
\begin{aligned}
\hat{c} \quad &\ge \quad \frac{\delta a(x)}{t} + \frac{\delta_1 a(x) S(x)}{t^2} - \frac{2S(x)}{t^3} \\
&= \quad \frac{\delta a(x)}{t} + \frac{S(x)(\delta_1 a(x) - 2/t)}{t^2},
\end{aligned}
$$

where $\delta_1 = \delta/(2m(a))$. Clearly we can assume that $|x| \le t+R$ in these estimates. From $a(x) \ge a_0(1+|x|)^{-\alpha}$, with $\alpha \in [0,1)$, we conclude that $\delta_1 a(x) - 2/t > 0$ for sufficiently large time t. Hence the entire coefficient $\hat{c} > 0$.

The proof of $\partial_t \hat{c} < 0$ is similar, since $-t\partial_t \hat{c}$ and $\hat{c}$ have identical leading coefficients of t^{-1} and proportional coefficients of t^{-2}.

(ii) Using definitions (4.2) and Lemma 4.3, we find that

$$\begin{aligned} -\partial_t w_1 + w &= \left(-\frac{\partial_t w_1}{w_1} + \frac{w}{w_1}\right) w_1 \\ &\ge \left(\frac{m-1}{t} - \frac{4}{3}\frac{S(x)}{t^2} + \frac{8}{t} + \frac{4}{3}\frac{S(x)}{t^2}\right) w_1 \\ &\ge 0. \end{aligned}$$

(iii) We can estimate the first factor on the left side by condition *(ii)*:

$$\begin{aligned} -\partial_t w_1 + 2w &= (-\partial_t w_1 + w) + w \\ &\ge w. \end{aligned}$$

The second factor needs more work. We rewrite

$$-\partial_t w_1 + 2aw_1 + 4w_1\partial_t \ln w - 2w = \left(-\frac{\partial_t w_1}{w_1} + 2a\right) w_1 + \left(4\frac{\partial_t w}{w} - 2\frac{w}{w_1}\right) w_1$$

and apply Lemma 4.3 to estimate $-\partial_t w_1/w_1$:

$$\begin{aligned} -\partial_t w_1 + 2aw_1 + 4w_1\partial_t \ln w - 2w \ \ge\ & \left(\frac{m-1}{t} - \frac{4}{3}\frac{S(x)}{t^2} + 2a(x)\right) w_1 \\ & + \left(-\frac{4m}{t} + \frac{4S(x)}{t^2} - \frac{16}{t} - \frac{8}{3}\frac{S(x)}{t^2}\right) w_1. \end{aligned}$$

Regrouping terms on the left side, we obtain

$$\begin{aligned} \partial_t w_1 + 2aw_1 + 4w_1\partial_t \ln w - 2w &\ge \left(-\frac{3m+17}{t} + 2a(x)\right) w_1 \\ &\ge a(x) w_1 \end{aligned}$$

for sufficiently large t; the latter inequality follows from

$$a(x) \ge a_0(1+t+R)^{-\alpha} \ge (3m+17)t^{-1}$$

whenever $|x| \le t+R$ and t is sufficiently large.

Thus we multiply the above lower bounds of the two factors in *(iii)*:

$$\begin{aligned} (-\partial_t w_1 + 2w)(-\partial_t w_1 + 2aw_1 + 4w_1\partial_t \ln w - 2w) &\ge w \cdot aw_1 \\ &= \frac{4a(x)(6t+S(x))}{3t^2} w_1^2 \\ &\ge \frac{4}{3}\frac{a(x)S(x)}{t^2} w_1^2, \end{aligned} \tag{4.4}$$

where we use $w/w_1 = 4(6t+S(x))/(3t^2)$ from definition (4.2).

It remains to bound the right side of inequality (iii). Since $\nabla w_1/w_1 = \nabla \ln w_1$, we calculate

$$\begin{aligned}\nabla w_1 - 2w_1\nabla \ln w &= (\nabla \ln w_1 - 2\nabla \ln w)\, w_1\\ &= \left(-\frac{\nabla S(x)}{6t+S(x)} + \frac{\nabla S(x)}{t}\right) w_1\\ &= \frac{(5t+S(x))\nabla S(x)}{t(6t+S(x))} w_1\end{aligned}$$

and form its square

$$(\nabla w_1 - 2w_1\nabla \ln w)^2 = \frac{(5t+S(x))^2|\nabla S(x)|^2}{t^2(6t+S(x))^2} w_1^2.$$

Condition $(S3)$ shows that $|\nabla S(x)|^2 \le a(x)S(x)$, so the above expression can be compared with the lower bound (4.4):

$$\begin{aligned}\frac{(5t+S(x))^2|\nabla S(x)|^2}{t^2(6t+S(x))^2} w_1^2 &\le \frac{a(x)S(x)(5t+S(x))^2}{t^2(6t+S(x))^2} w_1^2\\ &\le \frac{a(x)S(x)}{t^2} w_1^2.\end{aligned}$$

Hence condition (iii) holds.

(iv) Property $(S2)$ in Lemma 4.1 implies

$$\frac{w(t,x)}{w_1(t,x)} = \frac{4}{3}\frac{6t+S(x)}{t^2} \le Ct^{-\alpha}$$

for $|x| \le t+R$. This is equivalent to condition (iv).

(v) Formulas (4.3) show that $w_t \ge -mt^{-1}w$, which is stronger than (v).

(vi) It is convenient to rewrite this conditions as

$$\frac{w_1}{w}\left((\partial_t w)^2 + (\nabla w)^2\right) \le Ca(x)$$

and express it in terms of m and $S(x)$:

$$\frac{t^2}{6t+S(x)}\left(\left(-\frac{m}{t} + \frac{S(x)}{t^2}\right)^2 + \frac{|\nabla S(x)|^2}{t^2}\right) \le Ca(x).$$

Recall the estimates $S(x) = O(|x|^{2-\alpha})$ and $|\nabla S(x)|^2 \le a(x)S(x)$. Hence the left side is bounded by

$$\begin{aligned}\frac{t^2}{6t+S(x)}\left(\frac{2m^2}{t^2} + \frac{2S^2(x)}{t^4} + \frac{a(x)S(x)}{t^2}\right) &\le \frac{2m^2}{6t} + \frac{2S(x)}{t^2} + a(x)\\ &\le Ca(x)\end{aligned}$$

for sufficiently large $t \ge t_0$. The proof of Proposition 4.4 is complete.

5. Proof of the Main Theorem and Corollaries

It follows from Proposition 4.4 that w and w_1, defined by (4.1) and (4.2), are admissible weights for the decay estimates in Proposition 3.4. As a consequence we readily derive the main result and its corollaries.

Proof of Theorem 1.1. The estimates are trivial for small t, so we assume that t is large throughout our proof. We apply Proposition 3.4 with the weights defined by (4.1) and (4.2). For every $\delta > 0$,

$$\begin{aligned} \int e^{(m(a)-\delta)\frac{A(x)}{t}} a(x) u^2 \, dx &\leq C(N_0 + E_0) t^{2\delta - m(a)}, \\ \int e^{(m(a)-\delta)\frac{A(x)}{t}} \left(\frac{1}{t} + \frac{A(x)}{t^2}\right)^{-1} (u_t^2 + |\nabla u|^2) \, dx &\leq C(N_0 + E_0) t^{2\delta - m(a)}, \end{aligned}$$

where $t \geq t_0$ and t_0 is a sufficiently large number depending on a, n, and δ. To simplify the second estimate, we notice that

$$\left(\frac{1}{t} + \frac{A(x)}{t^2}\right)^{-1} = t\left(1 + \frac{A(x)}{t}\right)^{-1} \geq cte^{-\delta\frac{A(x)}{t}}$$

for sufficiently large $t \geq t_0$. The estimates in Theorem 1.1 follow with a loss of decay 2δ if we also observe that

$$N_0 + E_0 \leq C(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2),$$

where C depends on R, a, and n. We obtain the final form after replacing 2δ by δ.

Proof of Corollary 1.2. We add the two estimates in Theorem 1.1 and restrict the integration to $\{x : A(x) \geq t^{1+\epsilon}\}$:

$$\int_{A(x)\geq t^{1+\epsilon}} e^{(m(a)-\delta)\frac{A(x)}{t}} (au^2 + u_t^2 + |\nabla u|^2) \, dx \leq C_\delta(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2) t^{\delta - m(a)}.$$

From $A(x)/t \geq t^\epsilon$ and $a(x) \geq a_0(1 + t + R)^{-\alpha}$, we have that

$$\int_{A(x)\geq t^{1+\epsilon}} (u^2 + u_t^2 + |\nabla u|^2) \, dx \leq C_\delta(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2) t^{\delta + \alpha - m(a)} e^{-(m(a)-\delta)t^\epsilon}.$$

A slight increase of δ allows us to include the power of t in the exponential term. This completes the proof.

Proof of Corollary 1.4. To obtain the weighted energy estimate we combine Theorem 1.1 with the lower bound $(A1)$ from Proposition 1.3, namely

$$A_0(1 + |x|)^{2-\alpha} \leq A(x) \leq A_1(1 + |x|)^{2-\alpha}, \quad x \in \mathbf{R}^n.$$

The weighted L^2-norm requires more work. First Theorem 1.1 yields

$$\int e^{A_0(m(a)-\delta)\frac{|x|^{2-\alpha}}{t}} a(x) u^2 \, dx \leq C_\delta(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2) t^{\delta - m(a)}. \tag{5.1}$$

The upper bound on $A(x)$ shows the existence of $C > 0$, such that

$$\begin{aligned} a(x) &\geq C(A(x))^{-\frac{\alpha}{2-\alpha}} \\ &= Ct^{-\frac{\alpha}{2-\alpha}} \left(\frac{A(x)}{t}\right)^{-\frac{\alpha}{2-\alpha}} \\ &\geq Ct^{-\frac{\alpha}{2-\alpha}} e^{-\delta\frac{A(x)}{t}}, \end{aligned} \tag{5.2}$$

whenever $t \geq t_0$ is sufficiently large. We complete the proof by substituting this lower bound of $a(x)$ into inequality (5.1).

Proof of Corollary 1.5. This result is similar to but more precise than the second corollary. Using properties $(A3)$ and $(A4)$ in Proposition 1.3, we write the main decay estimates as

$$\begin{aligned}\int e^{(\frac{n-\alpha}{2-\alpha}-\delta)\frac{A(x)}{t}} a(x)u^2\,dx &\leq C_\delta(\|\nabla u_0\|_{L^2}^2+\|u_1\|_{L^2}^2)t^{\delta-\frac{n-\alpha}{2-\alpha}},\\ \int e^{(\frac{n-\alpha}{2-\alpha}-\delta)\frac{A(x)}{t}}(u_t^2+|\nabla u|^2)\,dx &\leq C_\delta(\|\nabla u_0\|_{L^2}^2+\|u_1\|_{L^2}^2)t^{\delta-\frac{n-\alpha}{2-\alpha}-1},\end{aligned}$$

where

$$A(x)\sim\frac{a_2}{(2-\alpha)(n-\alpha)}|x|^{2-\alpha},\quad |x|\to\infty.$$

Hence there exists $C>0$, such that

$$A(x)+C\geq\frac{a_2}{(2-\alpha+\delta)(n-\alpha)}|x|^{2-\alpha},\quad x\in\mathbf{R}^n.$$

The remaining part is a lower bound of $a(x)$ which repeats (5.2).

Appendix A. Weighted Energy Identities

We will verify Proposition 2.1 using simpler notations. Actually we will consider a more general non-homogeneous perturbation of the wave equation:

$$v_{tt}-\Delta v+Av_t+B\cdot\nabla v+Cv=h \tag{A.1}$$

with coefficients A, $B=(B_1,B_2,\ldots,B_n)$, and C in $C^1((0,\infty)\times\mathbf{R}^n)$. It is convenient to assume that

$$v(t,x)=0\ \text{ and }\ h(t,x)=0\ \text{ if }\ |x|>t+R,$$

for some $R>0$, and

$$v\in C((0,\infty),H^2(\mathbf{R}^n))\cap C^1((0,\infty),H^1(\mathbf{R}^n)),\quad h\in C((0,\infty),L^2(\mathbf{R}^n)).$$

Given functions P and Q in $C^2((0,\infty)\times\mathbf{R}^n)$, we multiply (A.1) with Pv_t+Qv :

$$(v_{tt}-\Delta v+Av_t+B\cdot\nabla v+Cv)(Pv_t+Qv)=h(Pv_t+Qv)$$

The left side can be transformed through integration by parts over $\mathbf{R}^n$. There are many terms, unfortunately, so we split the work into five steps.

Step 1. Multiply (A.1) by v_t and rearrange the terms on the left side:

$$\left(\frac{v_t^2+|\nabla v|^2}{2}\right)_t-\nabla\cdot(v_t\nabla v)+Av_t^2+v_tB\cdot\nabla v+C\left(\frac{v^2}{2}\right)_t=hv_t,$$

where the dot product means $x\cdot y=x_1y_1+\cdots x_ny_n$.

Step 2. Multiply (A.1) by v and rearrange the terms on the left side:

$$(v_tv)_t-\nabla\cdot(v\nabla v)-v_t^2+|\nabla v|^2+A\left(\frac{v^2}{2}\right)_t+B\cdot\nabla\frac{v^2}{2}+Cv^2=hv.$$

Step 3. Multiply the identity from Step 1 by P and integrate by parts over $\mathbf{R}^n$.

$$\begin{aligned}&\frac{d}{dt}\int P\frac{v_t^2+|\nabla v|^2}{2}\,dx-\int P_t\frac{v_t^2+|\nabla v|^2}{2}\,dx+\int\nabla P\cdot v_t\nabla v\,dx\\ &+\int(PAv_t^2+v_tPB\cdot\nabla v)\,dx+\frac{d}{dt}\int PC\frac{v^2}{2}\,dx-\int(PC)_t\frac{v^2}{2}\,dx\\ &=\int hPv_t\,dx.\end{aligned}$$

Notice that boundary terms vanish since the support of $v(t,\cdot)$ is compact.

Step 4. Multiply the identity from Step 2 by Q and integrate by parts over $\mathbf{R}^n$.

$$\begin{aligned}
&\frac{d}{dt}\int Qv_t v\ dx - \int Q_t v_t v\ dx + \int \nabla Q\cdot v\nabla v\ dx\\
&+\int (Q|\nabla v|^2 - Qv_t^2)\ dx + \frac{d}{dt}\int QA\frac{v^2}{2}\ dx - \int (QA)_t\frac{v^2}{2}\ dx\\
&-\int \nabla\cdot(QB)\frac{v^2}{2}\ dx + \int QCv^2\ dx\\
&=\int hQv\ dx.
\end{aligned}$$

In the final form we rewrite

$$Q_t v_t v = \frac{1}{2}(Q_t v^2)_t - \frac{1}{2}Q_{tt}v^2, \quad \nabla Q\cdot v\nabla v = \frac{1}{2}\nabla\cdot(v^2\nabla Q) - \frac{1}{2}v^2\Delta Q,$$

so the above integral identity becomes

$$\begin{aligned}
&\frac{d}{dt}\int (Qv_t v - \frac{1}{2}Q_t v^2)\ dx + \int \frac{1}{2}Q_{tt}v^2\ dx - \int \frac{1}{2}\Delta Q v^2\ dx\\
&+\int (Q|\nabla v|^2 - Qv_t^2)\ dx + \frac{d}{dt}\int QA\frac{v^2}{2}\ dx - \int (QA)_t\frac{v^2}{2}\ dx\\
&-\int \nabla\cdot(QB)\frac{v^2}{2}\ dx + \int QCv^2\ dx\\
&=\int hQv\ dx.
\end{aligned}$$

Step 5 Add the final identities from Step 3 and Step 4 and combine terms with the same order of derivatives. This leads us to the following result.

Proposition A.1. *Let v be the solution of (A.1). If P and Q are C^2-functions,*

$$\frac{d}{dt}E(v_t,\nabla v,v) + F(v_t,\nabla v) + G(v) = H(h,v_t,v),$$

where the functionals are given by

$$E(v_t,\nabla v,v) = \frac{1}{2}\int [P(v_t^2+|\nabla v|^2) + 2Qv_t v + (CP - Q_t + AQ)v^2]\ dx,$$

$$\begin{aligned}
F(v_t,\nabla v) &= \frac{1}{2}\int (-P_t + 2AP - 2Q)v_t^2\ dx\\
&\quad+\int (\nabla P + BP)\cdot v_t\nabla v\ dx\\
&\quad+\frac{1}{2}\int (-P_t + 2Q)|\nabla v|^2\ dx,
\end{aligned}$$

$$G(v) = \frac{1}{2}\int [Q_{tt} - \Delta Q - (AQ)_t - \nabla\cdot(QB) + 2CQ - (CP)_t]v^2\ dx,$$

$$H(h,v_t,v) = \int h(Pv_t + Qv)\ dx.$$

Proof of Proposition 2.1. The result is a special case of Proposition A.1. Let the functions be $v = \hat{u}$ and $h = 0$. We obtain the identity in Proposition 2.1 if

$$\begin{aligned} &A = \hat{a}, \quad B = \hat{b}, \quad C = \hat{c}, \\ &P = w_1, \quad Q = w_0, \end{aligned}$$

are the coefficients and weights, respectively.

Appendix B. The Poisson Equation in $\mathbf{R}^n$

We will study the positive radial solutions of

$$\Delta A(x) = a(x), \quad x \in \mathbf{R}^n, \tag{B.1}$$

for positive radial $a(x)$ in $n \geq 1$ dimensions. (When $n = 1$ we consider even functions $A(x)$ and $a(x)$.) The main problem is to show that there exists a radial solution in the class of functions satisfying $(a1) - (a3)$. Additional difficulties are to find the asymptotic behavior of $A(x)$ as $|x| \to \infty$ and compute the limit $m(a)$. We will find three different representations of $A(x)$ depending on whether $n = 1$, $n = 2$, or $n \geq 3$.

When $n = 1$, we have

$$A(r) \;=\; 1 + \int_0^r (r - \tau) a(\tau)\, d\tau, \quad r \in \mathbf{R}. \tag{B.2}$$

However, we can consider only non-negative r since $A(r)$ is an even function.

When $n \geq 2$, we rely on radial symmetry to simplify the Poisson equation:

$$\frac{d^2 A}{dr^2} + \frac{n-1}{r}\frac{dA}{dr} = a(r), \quad r = |x|,$$

where we write $a(r)$ instead of $a(r\omega)$, $\omega \in \mathbf{S}^{n-1}$. We multiply with r^{n-1} and transform the equation into

$$\frac{d}{dr}\left(r^{n-1}\frac{dA}{dr}\right) = r^{n-1} a(r).$$

Integrating on $[0, r]$ yields

$$\frac{dA(r)}{dr} = r^{1-n}\int_0^r \tau^{n-1} a(\tau)\, d\tau. \tag{B.3}$$

If we repeat the integration and choose $A(0) = 1$, the resulting solution is

$$A(r) \;=\; 1 + \int_0^r \left(\ln\frac{r}{\tau}\right) \tau a(\tau)\, d\tau, \quad n = 2, \tag{B.4}$$

$$A(r) \;=\; 1 + \frac{1}{n-2}\int_0^r \left(1 - \frac{\tau^{n-2}}{r^{n-2}}\right) \tau a(\tau)\, d\tau, \quad n \geq 3. \tag{B.5}$$

Notice that the solutions of Poisson equation given by (B.2), (B.4), and (B.5) satisfy $A(0) = 1$ and $dA(0)/dr = 0$. When $n \geq 2$, $A(r)$ means $A(r\omega)$ with $\omega \in \mathbf{S}^{n-1}$.

The above formulas for $A(r)$ lead to the following two-sided bounds.

Proposition B.1. *Assume that $a(x)$ is an even function, if $n = 1$, or that $a(x)$ depends only on $r = |x|$, if $n \geq 2$. If $a \in C(\mathbf{R}^n)$ and*

$$a_0(1 + |x|)^{-\alpha} \leq a(x) \leq a_1(1 + |x|)^{-\alpha}, \quad \alpha \in [0, 1),$$

where a_0 and a_1 are positive constants, the solution of equation (B.1) defined in (B.2), (B.4), or (B.5) satisfies $A(0) = 1$, $dA(0)/dr = 0$, and

$$A_0(1+|x|)^{2-\alpha} \le A(x) \le A_1(1+|x|)^{2-\alpha}, \qquad x \in \mathbf{R}^n,$$
$$A_2(1+|x|)^{1-\alpha} \le |\nabla A(x)| \le A_3(1+|x|)^{1-\alpha}, \quad |x| \ge 1,$$

for some constants $A_0, \dots, A_3 > 0$.

Proof. We can write a common formula

$$A(r) \quad = \quad 1 + r^2 \int_0^1 K_n(\tau) a(r\tau)\, d\tau, \quad r > 0, \tag{B.6}$$

where

$$K_n(\tau) = \begin{cases} 1-\tau, & \text{if } \ n = 1, \\ \tau \ln \frac{1}{\tau}, & \text{if } \ n = 2, \\ \frac{1}{n-2}\tau(1-\tau^{n-2}), & \text{if } \ n \ge 3. \end{cases}$$

It is easy to see that there exist $L_n > l_n > 0$, such that

$$K_n(\tau) \le L_n, \quad \tau \in [0,1] \quad \text{ and } \quad K_n(\tau) \ge l_n, \quad \tau \in [1/4, 1/2].$$

From these bounds, formula (B.6), and our assumptions on a, we obtain

$$1 + a_0 l_n I_0(r) r^2 \le A(r) \le 1 + a_1 L_n I_1(r) r^2, \quad r > 0, \tag{B.7}$$

with

$$I_k(r) = \int_{(1-k)/4}^{1/(2-k)} (1 + r\tau)^{-\alpha}\, d\tau, \quad k = 0, 1.$$

Now the trivial estimates

$$\begin{aligned} I_0(r) \quad &\ge \quad \int_{1/4}^{1/2} (1+r)^{-\alpha}\, d\tau = (1+r)^{-\alpha}/4, \\ I_1(r) \quad &\le \quad \int_0^1 (r\tau)^{-\alpha}\, d\tau = r^{-\alpha}/(1-\alpha), \end{aligned}$$

and (B.7) yield the two-sided bound on $A(r)$ in Proposition B.1.

To show the second claim of Proposition B.1, we use

$$\frac{dA(r)}{dr} \quad = \quad r \int_0^1 \tau^{n-1} a(r\tau)\, d\tau, \quad r > 0,$$

which is valid in any dimension $n \ge 1$; see (B.2) and (B.3). Clearly the assumptions on a imply

$$1 + a_0 J_0(r) r \le \frac{dA(r)}{dr} \le 1 + a_1 J_1(r) r, \quad r > 0, \tag{B.8}$$

with

$$J_k(r) = \int_{(1-k)/2}^1 \tau^{n-1} (1 + r\tau)^{-\alpha}\, d\tau, \quad k = 0, 1.$$

The simple estimates

$$\begin{aligned} J_0(r) \quad &\ge \quad (1+r)^{-\alpha}/2^n, \\ J_1(r) \quad &\le \quad r^{-\alpha}/(1-\alpha), \end{aligned}$$

and (B.8) complete the proof of Proposition B.1.

Proof of Proposition 1.3, claims ($A1$) and ($A2$). We apply Proposition B.1 to the solution A given by (B.5). Clearly this solution meets conditions ($A1$)–($A2$).

Proof of Proposition 1.3, claims ($A3$) and ($A4$). It follows from (B.3) and the asymptotics $a(r) = a_2 r^{-\alpha} + o(r^{-\alpha})$ that

$$\frac{dA(r)}{dr} = r^{1-n} \int_0^r \tau^{n-1} [a_2 \tau^{-\alpha} + o(\tau^{-\alpha})] \, d\tau, \quad r \to \infty.$$

Hence

$$\begin{aligned} \frac{dA(r)}{dr} &= \frac{a_2}{n-\alpha} r^{1-\alpha} + o(r^{1-\alpha}), \\ A(r) &= \frac{a_2}{(n-\alpha)(2-\alpha)} r^{2-\alpha} + o(r^{2-\alpha}), \end{aligned}$$

as $r \to \infty$. A substitution into $m(a)$ shows that

$$m(a) = \frac{n-\alpha}{2-\alpha}.$$

The proof of Proposition 1.3 is complete.

References

[AL] Asch, Mark; Lebeau, Gilles, *The spectrum of the damped wave operator for a bounded domain in R^2*, Experiment. Math. 12 (2003), no. 2, 227–241.

[BN] Bae, Jeong Ja; Nakao, Mitsuhiro, *Energy decay for the wave equation with boundary and localized dissipations in exterior domains*, Math. Nachr. 278 (2005), no. 7-8, 771–783.

[CZ] Cox, Steven; Zuazua, Enrique, *The rate at which energy decays in a damped string*, Comm. Partial Differential Equations 19 (1994), no. 1-2, 213–243.

[DC] Dafermos, C. M., *Asymptotic behavior of solutions of evolution equations*, Nonlinear evolution equations (Proc. Sympos., Univ. Wisconsin, Madison, Wis., 1977), pp. 103–123, Publ. Math. Res. Center Univ. Wisconsin, 40, Academic Press, New York-London, 1978.

[DG] Dehman, B; Gérard, P., *Stabilization for the Nonlinear Klein Gordon Equation with Critical Exponent*, Preprint (2002).

[DLZ] Dehman, Belhassen; Lebeau, Gilles; Zuazua, E., *Stabilization and control for the subcritical semilinear wave equation*, Ann. Sci. cole Norm. Sup. 36 (2003), no. 4, 525–551.

[FJ] Ferreira, Jos da Silva, *Exponential decay for a nonlinear system of hyperbolic equations with locally distributed dampings*, Nonlinear Anal. 18 (1992), no. 11, 1015–1032.

[FZ] Freitas, Pedro; Zuazua, Enrique, *Stability results for the wave equation with indefinite damping*, J. Differential Equations 132 (1996), no. 2, 338–352.

[HA] Haraux, Alain, *Stabilization of trajectories for some weakly damped hyperbolic equations*, J. Differential Equations 59 (1985), no. 2, 145–154.

[HM] Hitrik, Michael, *Eigenfrequencies and expansions for damped wave equations*, Methods Appl. Anal. 10 (2003), no. 4, 543–564.

[IM1] Ikehata, Ryo; Matsuyama, Tokio, *Remarks on the behaviour of solutions to the linear wave equations in unbounded domains*, Proc. Sch. Sci. Tokai Univ. 36 (2001), 1–13.

[IM2] Ikehata, Ryo *Fast decay of solutions for linear wave equations with dissipation localized near infinity in an exterior domain*, J. Differential Equations 188 (2003), no. 2, 390–405.

[I1] Ikehata, Ryo, *New decay estimates for linear damped wave equations and its application to nonlinear problem*, Math. Methods Appl. Sci. 27 (2004), no. 8, 865–889.

[I2] Ikehata, Ryo, *Some remarks on the wave equation with potential type damping coefficients*, Int. J. Pure Appl. Math. 21 (2005), no. 1, 19–24.

[LT] Lasiecka, Irena; Toundykov, Daniel, *Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms*, Nonlinear Anal. 64 (2006), no. 8, 1757–1797.

[KNS] Kawashita, Mishio; Nakazawa, Hideo; Soga, Hideo, *Non decay of the total energy for the wave equation with the dissipative term of spatial anisotropy*, Nagoya Math. J. 174 (2004), 115–126.

[KV] Komornik, Vilmos, *Decay estimates for the wave equation. Control and optimal design of distributed parameter systems*, (Minneapolis, MN, 1992), 153–169, IMA Vol. Math. Appl., 70, Springer, New York, 1995.

[MMR] Marcati, Pierangelo; Mei, Ming; Rubino, Bruno, *Optimal convergence rates to diffusion waves for solutions of the hyperbolic conservation laws with damping*, J. Math. Fluid Mech. 7 (2005), suppl. 2, S224–S240.

[MP] Martinez, Patrick, *Decay of solutions of the wave equation with a local highly degenerate dissipation*, Asymptot. Anal. 19 (1999), no. 1, 1–17.

[M1] Matsumura, Akitaka, *On the asymptotic behavior of solutions of semi-linear wave equations*, Publ. Res. Inst. Math. Sci. 12 (1976/77), no. 1, 169–189.

[M2] Matsumura, Akitaka, *Energy decay of solutions of dissipative wave equations*, Proc. Japan Acad. Ser. A Math. Sci. 53 (1977), no. 7, 232–236.

[MY1] Milani, Albert; Yang, Han *On the diffusion phenomenon of quasilinear hyperbolic waves in low space dimensions*, Bull. Sci. Math. 124 (2000), 415–433.

[MY2] Milani, Albert; Yang, Han L^1 *decay estimates for dissipative wave equations*, Math. Methods Appl. Sci. 24 (2001), no. 5, 319–338.

[MK] Mochizuki, K., *Scattering theory for wave equations with dissipative terms*, Publ. Res. Inst. Math. Sci. 12 (1976/77), no. 2, 383–390.

[MN] Mochizuki, Kiyoshi; Nakao, Mitsuhiro, *Total energy decay for the wave equation in exterior domains with a dissipation near infinity*, J. Math. Anal. Appl. 326 (2007), no. 1, 582–588.

[MN1] Mochizuki, K. and Nakazawa, H., *Energy decay and asymptotic behavior of solutions to the wave equations with linear dissipation*, Publ. Res. Inst. Math. Sci. 32 (1996), no. 3, 401–414.

[MN2] Mochizuki, Kiyoshi; Nakazawa, Hideo, *Energy decay of solutions to the wave equations with linear dissipation localized near infinity*, Publ. Res. Inst. Math. Sci. 37 (2001), no. 3, 441–458.

[MC] Morawetz, C., *The decay of solutions of the exterior initial-boundary problem for the wave equation*, Comm. Pure Appl. Math. Sci. 14 (1961), 561–568.

[NM] Nakao, M., *Energy decay for the linear and semilinear wave equations in exterior domains with some localized dissipations*, Math. Z. 238 (2001), no. 4, 781–797.

[NT1] Narazaki, Takashi, $L^p - L^q$ *estimates for damped wave equations and their applications to semi-linear problem*, J. Math. Soc. Japan 56 (2004), 585–626.

[NT2] Narazaki, Takashi, $L^p - L^q$ *estimates for damped wave equations with odd initial data*, Electron. J. Differential Equations 2005, No. 74, 17 pp. (electronic).

[NK1] Nishihara, Kenji, *Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping*, J. Diff. Eq. 137 (1997), 384–395.

[NK2] Nishihara, Kenji, $L^p - L^q$ *estimates of solutions to the damped wave equation in 3-dimensional space and their application*, Math. Z. 244 (2003), no. 3, 631–649.

[RT1] Rauch, Jeffrey; Taylor, Michael, *Decay of solutions to nondissipative hyperbolic systems on compact manifolds*, Comm. Pure Appl. Math. 28 (1975), no. 4, 501–523.

[RT2] Rauch, Jeffrey; Taylor, Michael, *Exponential decay of solutions to hyperbolic equations in bounded domains*, Indiana Univ. Math. J. 24 (1974), 79–86.

[RM] Reissig, Michael, $L^p - L^q$ *decay estimates for wave equations with time-dependent coefficients*, J. Nonlinear Math. Phys. 11 (2004), no. 4, 534–548.

[RW] Reissig, Michael; Wirth, Jens, $L^p - L^q$ *estimates for wave equations with monotone time-dependent dissipation*, Proceedings of the RIMS Symposium on MathematicalModels of Phenomena and Evolution Equations (to appear).

[SJ] Sjöstrand, Johannes, *Asymptotic distribution of eigenfrequencies for damped wave equations*, Publ. Res. Inst. Math. Sci. 36 (2000), no. 5, 573–611.

[SR] Strichartz, Robert, *A priori estimates for the wave equation and some applications*, J. Funct. Anal. 5 (1970), 218–235.

[TY] Todorova, Grozdena; Yordanov, Borislav, *Nonlinear dissipative wave equations with potential*, Control Mehtods in PDE-Dynamical Systems, 317–337, Contemp. Math., 426, Amer. Math. Soc., Providence, RI, 2007.

[UH] Uesaka, Hiroshi, *The total energy decay of solutions for the wave equation with a dissipative term*, J. Math. Kyoto Univ. 20 (1980), no. 1, 57–65.

[W1] Wirth, J., *Solution representations for a wave equation with weak dissipation*, Math. Methods Appl. Sci. 27 (2004), no. 1, 101–124.

[W2] Wirth, J., *Wave equations with time-dependent dissipation I. Non-effective dissipation*, J. Diff. Eq. 222, (2006), 487–514.

[W3] Wirth, J., *Wave equations with time-dependent dissipation. II. Effective dissipation*, J. Differential Equations 232 (2007), no. 1, 74–103.

[WZ] Wong, Bun; Zhang, Qi S., *Refined gradient bounds, Poisson equations and some applications to open Khler manifolds*, Asian J. Math. 7 (2003), no. 3, 337–364.

[Z1] Zuazua, E., *Exponential decay for the semilinear wave equation with locally distributed damping*, Comm. Partial Differential Equations 15 (1990), no. 2, 205–235.

[Z2] Zuazua, E., *Exponential decay for the semilinear wave equation with localized damping in unbounded domains*, J. Math. Pures Appl. 70 (1991), no. 4, 513–529.

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