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THE ENERGY DECAY PROBLEM FOR WAVE EQUATIONS WITH NONLINEAR DISSIPATIVE TERMS IN $\mathbb{R}^n$

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Abstract. We study the asymptotic behavior of energy for wave equations with nonlinear damping $g(u_t) = |u_t|^{m-1}u_t$ in $\mathbb{R}^n$ $(n \geq 3)$ as time $t \to \infty$. The main result shows a polynomial decay rate of energy under the condition $1 < m \leq (n+2)/(n+1)$. Previously, only logarithmic decay rates were found.

1. Introduction

The energy decay for dissipative equations of the form

$$u_{tt} - \Delta u + g(t, x, u_t) = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{1.1}$$

with initial data

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

is an important question and a key starting point for many open problems. Here the nonlinear dissipation satisfies $g \in C^1$ and $g(t, x, u_t)u_t \geq 0$. The global well-posedness is established by Lions and Strauss [LS] for initial data $u_0 \in H^{s+1}(\mathbb{R}^n)$, $u_1 \in H^s(\mathbb{R}^n)$, $\quad 0 \leq s \leq 1$.

When $s = 1$, a more precise result is available due to Liang [L], Motai [Mo], and Serrin, Todorova and Vitillaro [STV]. It is known that if $g(t, x, u_t) = |u_t|^{m-1}u_t$ where $m > 1$ and

$$u_0 \in H^2(\mathbb{R}^n), \quad u_1 \in H^1(\mathbb{R}^n) \cap L^{2m}(\mathbb{R}^n),$$

problem (1.1), (1.2) has a unique global solution u with the following properties:

(a) $u \in C(\mathbb{R}_+, H^2(\mathbb{R}^n)),\ u_t \in C(\mathbb{R}_+, H^1(\mathbb{R}^n)),\ u_{tt} \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^n))$.

(b) The energy identity holds:

$$E(t) = E(0) - \int_0^t \int g(s, x, u_s)u_s dxds,$$

where

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2)dx. \tag{1.3}$$

(c) The solution u has a finite speed of propagation.

It follows from the energy identity that $E(t)$ is a decreasing function of $t > 0$. A naturally arising question is *whether the energy $E(t)$ decays to zero or not as $t \to \infty$*.

Let us mention that the boundary value problem $x \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, is quite different from the Cauchy problem (1.1), (1.2). Nakao

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[N1] and Haraux [H] have found polynomial decay rates of $E(t)$ under the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. The current state of this problem and its generalizations for localized damping and source is presented in Lasiecka and Toundykov [LT] (see also the references therein).

Below is a summary of results on the asymptotic behavior of $E(t)$ for the Cauchy problem (1.1), (1.2).

1. For the linear dissipation

$$g(t, x, u_t) = u_t,$$

the exact polynomial decay $E(t) \leq Ct^{-n/2-1}$ is found by Matsumura [M].

2. For nonlinear dissipations $g(t, x, u_t) = |u_t|^{m-1}u_t$, a polynomial decay rate is derived in *the presence of a mass term in equation (1.1)* by Nakao [N2] (compactly supported data) and Mochizuki and Motai [MM]. These authors consider the nonlinear dissipative Klein-Gordon equation

$$u_{tt} - \Delta u + u + |u_t|^{m-1}u_t = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n \tag{1.4}$$

and show the following: if $1 < m < 1 + 2/n$, the energy decays according to

$$E(t) \leq C(1 + t)^{-\gamma}, \quad t \to \infty,$$

where $\gamma = 2/(m - 1) - n$; if $m > 1 + 2/n$, [MM] establishes a complementary non-decay result: there exists a dense set of initial data in $H^1 \times L^2$ for which $E(t)$ does not decay.

3. The *best known decay estimate* for equation (1.1) with power dissipations is due to Mochizuki and Motai [MM]. They show a *logarithmic decay rate* of energy for the equation

$$u_{tt} - \Delta u + |u_t|^{m-1}u_t = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{1.5}$$

with exponents satisfying $1 < m < 1 + 2/n$:

$$E(t) \leq C\{\ln(2 + t)\}^{-\gamma},$$

where $\gamma > 0$ depends on m. The corresponding non-decay result in [MM] requires $m > 1 + 2/(n - 1)$, although the case of equation (1.4) suggests that $m > 1 + 2/n$ might be sufficient.

It is expected, however, that the energy of (1.5) decays at a polynomial rate. The main difficulty in establishing such results seems to be the lack of control of the L^2 norm of the solution. This is an essential difference with the equation in a bounded domain or the Klein-Gordon equation. There are works improving the logarithmic decay rate for the sake of making the dissipation linear for large $|x|$. Nakao and Jung [NJ] consider a dissipation which is allowed to be nonlinear only in a ball, but *outside that ball the dissipation must be linear.* The linearity of the dissipation for large $|x|$ makes it possible to control the L^2 norm of the solution.

In this paper we find a polynomial decay of $E(t)$ for all solutions of (1.5) with compactly supported data $(u_0, u_1) \in H^2 \times H^1$. The main idea is to use the "parabolic" effects coming from the presence of the damping term. We show that the damping changes drastically the asymptotic behavior of solutions as follows.

1. *The energy localized in* $|x| > t^{(1+\delta)/2}$, *with* $\delta > 0$, *decays fast.*
Let u be a solution of equation (1.5) and let the exterior energy $E_{\text{ext}}(t)$ be

$$E_{\text{ext}}(t) = \frac{1}{2}\int_{|x|>t^{(1+\delta)/2}} (u_t^2 + |\nabla u|^2)dx, \tag{1.6}$$

for $\delta \in (0,1)$. Then we have the estimate

$$E_{\text{ext}}(t) \le C t^{n/2+1-((m+1)/(m-1)-n/2)\delta} \ln t, \quad t \to \infty.$$

The exponent of t shows that the decay of $E_{\text{ext}}(t)$ is fast when $m \approx 1$. Similarly, the decay of $E_{\text{ext}}(t)$ is slow when $m \approx 1+2/n$. No decay is expected if $m \ge 1+2/n$. This observation is consistent with the non-decay result on the Klein-Gordon equation (1.4) in [MM], since the wave equation (1.5) is expected to have slower energy decay.

The exterior energy can be studied by weighted estimates. The strongest parabolic effects are manifested in the case of linear damping, $m = 1$, which allows weights of the from $w(t,x) = e^{|x|^2/(2t)}$. In this case the exterior energy of (1.5) decays exponentially. Namely, the estimate of [TY] is

$$E_{\text{ext}}(t) \le C e^{-t^{2\delta}/2}, \quad t \to \infty.$$

A suitable weight for nonlinear dissipations is

$$w(t,x) = (1 + |x|^2/t)^{b/2}, \tag{1.7}$$

with exponent b depending on m and n. The idea to use such weights comes from the asymptotic behavior of fast diffusion equations

$$\partial_t v^m - \Delta v = 0.$$

In fact, every positive solution v has the asymptotic profile

$$v(t,x) \sim t^{-n\gamma}(\alpha + \beta|x|^2/t^{2m\gamma})^{-1/(m-1)}, \tag{1.8}$$

with positive constants α, β, and $\gamma = (n - m(n-2))^{-1}$ (see Carrillo and Vázquez [CV] and the references therein). The wave equation with a nonlinear damping (1.5) is formally transformed into the fast diffusion equation if $\partial_t^2 u$ is neglected, the remaining terms are differentiated with respect to t, and $\partial_t u$ is replaced by v. In general these manipulations can not be justified, although they are valid when the damping is linear; $\partial_t^2 u$ is much smaller than the other two terms in equation (1.5) as $t \to \infty$, see [M]. Thus we expect the phenomenon to persist when the damping is close to linear. The part $|x|^2/t^{2m\gamma}$ of the weight in (1.8) asymptotically approaches $|x|^2/t$ when $m \to 1$, since $\gamma \to 1/2$ as $m \to 1$. This explains the weight in (1.7).

The decay rate of the interior energy $E_{\text{int}}(t)$, defined as

$$E_{\text{int}}(t) \;=\; \frac{1}{2}\int_{|x| \le t^{(1+\delta)/2}} (u_t^2 + |\nabla u|^2)\,dx,$$

is much slower than the decay rate of the exterior energy $E_{\text{ext}}(t)$. This further restricts the decay rate of the total energy $E(t)$. Here we also observe a significant difference in the asymptotic behavior of u_t and ∇u.

2. *The scaling invariance of (1.5) implies a polynomial decay of* $\|u_t\|_{L^2}$. We show that

$$\|u_t\|_{L^2} \le C t^{\frac{(1+\delta)n}{4}\frac{m-1}{m+1} - \frac{1}{m+1}} \ln^{\frac{1}{2}} t, \quad t \to \infty,$$

where

$$\delta = \frac{(m-1)(m+n+3)}{(m+1)^2 - (m-1)n}.$$

It is easy to see that $\delta < 1$ if $m < 1 + \frac{2}{n}$. The result is based on the scaling invariance of equation (1.5). In fact, we have a weighted estimate of second order involving the scaling operator $S = t\partial_t + x \cdot \nabla_x$:

$$\frac{1}{2}\int ((Su)_t^2 + |\nabla Su|^2)dx \le C.$$

As a consequence, we can derive

$$\|u_t\|_{L^{m+1}} \le Ct^{\frac{1}{m+1}}$$

and, using the fast decay of $E_{\text{ext}}(t)$, the above estimate implies a decay of $\|u_t\|_{L^2}$. This is the only place where the higher regularity of initial data is essential. The other results hold for data in the energy space.

3. *Weighted L^p-estimate for $(\partial_t + \sqrt{-\Delta})^2$ help bound ∇u in terms of u_t.* We expect $\|\nabla u\|_{L^2}$ to have the slowest decay rate which determines the decay rate of the entire energy. This part is more technical. First we establish space-time L^p estimates of ∇u in terms of $\partial_t u$. To clarify the effects of dissipation, we rewrite equation (1.5) in the parabolic form

$$(\partial_t + \sqrt{-\Delta})^2 u = -|\partial_t u|^{m-1}\partial_t u + 2\sqrt{-\Delta}\partial_t u.$$

The final decay estimate is

$$\begin{aligned} &\int_0^t \int (s + |x|^2)^{\frac{a}{2} - \frac{m+1}{2m}} |\nabla u|^{\frac{m+1}{m}} dxds \\ \le\quad & F_m \int_0^t \int (s + |x|^2)^{\frac{a}{2}} |\partial_s u|^{m+1} dxds + G_{m,a}, \end{aligned}$$

where m and a satisfy

$$n + \frac{a}{2} - \frac{m+1}{2(m-1)} < -1 \quad \text{and} \quad 0 < a < 1.$$

A key feature of this result is that F_m depends on m but not on the parameter a. The other constant $G_{m,a}$ depends also on the initial data u_0 and u_1. It follows that the decay rates of ∇u and $\partial_t u$ are closely related, with the latter being slightly faster. Our proof is based on classical estimates for convolution operators in weighted L^p spaces. Similar results hold for all L^p norms of ∇u and can be used to study the regularizing effect of nonlinear dissipation. The details will be presented in a forthcoming paper.

Finally, we combine the space-time estimates of ∇u with weighted energy estimates to derive a polynomial decay rate for the energy.

Theorem 1.1. *Let u be a solution of (1.5) with compactly supported initial data $(u_0, u_1) \in H^2 \times H^1$. For $n \ge 3$ and $(n+2)/(n+1) \ge m > 1$, the energy (1.3) decays polynomially:*

$$E(t) \le Ct^{-a/2}, \quad t \to \infty,$$

where the positive exponent a depends only on m and n.

The paper is organized as follows. Section 2 deals with $E_{\text{ext}}(t)$. Section 3 contains the decay estimates of $\|u_t\|_{L^2}$. Section 4 is devoted to the space-time L^p estimates for ∇u. The main result about the polynomial decay of $E(t)$ is shown in Section 5. Finally we discuss some open problems in Section 6. In the Appendix we present basic facts from harmonic analysis which are used only in Section 4.

2. Decay Rate of $E_{\text{ext}}(t)$

In this section we show a weighted energy estimate which implies that $E_{\text{ext}}(t)$ decays fast as $t \to \infty$ if the damping is close to linear ($m \approx 1$).

Proposition 2.1. *Let u be a solution of equation (1.5) with compactly supported initial data $(u_0, u_1) \in H^1 \times L^2$. Define the external energy $E_{\text{ext}}(t)$ by (1.6). Then*

$$E_{\text{ext}}(t) \leq C t^{n/2+1-((m+1)/(m-1)-n/2)\delta} \ln t, \quad t \geq 2,$$

for any $n \geq 1$ and $\delta \in (0,1)$.

We give the proof after deriving a weighted energy identity. For a positive $w \in C^1$, we multiply equation (1.5) with wu_t and rearrange the terms to obtain

$$\begin{aligned} & \left(\frac{w}{2}(u_t^2 + |\nabla u|^2)\right)_t - \operatorname{div}(wu_t \nabla u) \\ = \;& -w|u_t|^{m+1} + \frac{w_t}{2}(u_t^2 + |\nabla u|^2) + \nabla w \cdot u_t \nabla u. \end{aligned}$$

Useful forms of w are introduced later. Since $u(t, \cdot)$ is compactly supported, we can integrate over $\mathbb{R}^n$ and use the divergence theorem. Then

$$\begin{aligned} & \frac{d}{dt} \int \frac{w}{2}(u_t^2 + |\nabla u|^2) dx \\ = \;& -\int w|u_t|^{m+1} dx + \int \frac{w_t}{2}(u_t^2 + |\nabla u|^2) dx + \int \nabla w \cdot u_t \nabla u \, dx. \end{aligned} \tag{2.1}$$

This is the main identity. It plays a crucial role in both the current section and Section 6, with a different weight w.

Proof of Proposition 2.1. We can assume that $t \geq 2$ and $|x| > 0$. Let

$$w(t,x) = (1 + |x|^2/t)^{b/2},$$

where the optimal $b > 0$ is to be chosen. To estimate the right side of identity (2.1), we compute the derivatives

$$\begin{aligned} w_t &= -b\frac{|x|^2}{t^2}(1 + |x|^2/t)^{b/2-1}, \\ \nabla w &= b\frac{x}{t}(1 + |x|^2/t)^{b/2-1}, \end{aligned}$$

and the quotient

$$\frac{|\nabla w|^2}{-w_t w} = b(1 + |x|^2/t)^{-1}.$$

We will bound $\nabla w \cdot u_t \nabla u$ by two applications of Young's inequality using $w_t < 0$:

$$\begin{aligned} |\nabla w \cdot u_t \nabla u| &\leq \frac{1}{2}(-w_t)|\nabla u|^2 + \frac{1}{2}\frac{|\nabla w|^2}{-w_t} u_t^2 \\ &\leq -\frac{1}{2} w_t |\nabla u|^2 + \frac{1}{m+1} w|u_t|^{m+1} + \frac{m-1}{2(m+1)} \left(\frac{|\nabla w|^2}{-w_t w}\right)^{\frac{m+1}{m-1}} w\chi, \end{aligned}$$

where χ is the characteristic function of the ball $\{x : |x| \le t + R\}$. It follows from this estimate and identity (2.1) that

$$\frac{d}{dt}\int \frac{w}{2}(u_t^2 + |\nabla u|^2)dx \quad \le \quad -\frac{m}{m+1}\int w|u_t|^{m+1}dx + \int \frac{w_t}{2}|\nabla u|^2 dx$$
$$+\frac{m-1}{2(m+1)}\int \left(\frac{|\nabla w|^2}{-w_t w}\right)^{\frac{m+1}{m-1}} w\chi dx.$$

Neglecting the two negative terms, we have a simpler bound:

$$\frac{d}{dt}\int \frac{w}{2}(u_t^2 + |\nabla u|^2)dx \quad \le \quad \frac{m-1}{2(m+1)}\int \left(\frac{|\nabla w|^2}{-w_t w}\right)^{\frac{m+1}{m-1}} w\chi dx$$
$$= \quad C\int_{|x|\le t+R} (1 + |x|^2/t)^{\frac{b}{2}-\frac{m+1}{m-1}} dx.$$

If $b/2 - (m+1)/(m-1) \le -n/2$, the last integral is not greater than

$$Ct^{n/2}\int_{|y|\le\sqrt{t}+R} (1 + |y|^2)^{-n/2}dy \le Ct^{n/2}\ln t,$$

which means that

$$\frac{d}{dt}\int (1 + |x|^2/t)^{b/2}(u_t^2 + |\nabla u|^2)dx \quad \le \quad Ct^{n/2}\ln t.$$

Integrating over $[2, t]$, we have the weighted estimate

$$\int (1 + |x|^2/t)^{b/2}(u_t^2 + |\nabla u|^2)dx \quad \le \quad Ct^{n/2+1}\ln t, \quad t \ge 2,$$

for all $b \le 2(m+1)/(m-1) - n$. The result implies fast decay of the local energy in region where the quotient $|x|^2/t$ is large. In particular, we have

$$(1 + |x|^2/t)^{b/2} \ge t^{b\delta/2} \quad \text{if} \quad |x| \ge t^{(1+\delta)/2}.$$

This completes the proof of of Proposition 2.1. □

It is interesting to consider the two limit cases. If the exponent is $m \approx 1$, we have that $E_{\text{ext}}(t)$ decays faster than any power of t. This is consistent with our knowledge of the linear case $m = 1$, in which the external energy decays exponentially.

The upper bound on admissible exponents m is determined from the condition $\delta < 1$. In fact, $\delta = 1$ means that the support is no longer suppressed inside a small subset of the ball $|x| \le t + R$. Proposition 2.1 shows that $E_{\text{ext}}(t)$ with $\delta = 1$ will decay as long as

$$\frac{n}{2} + 1 - \frac{m+1}{m-1} + \frac{n}{2} > 0.$$

Solving for m, we obtain the condition $m < 1 + 2/n$.

3. Decay Rate of $\|u_t\|_{L^2}$

The invariance of equation (1.5) under scaling transforms allows weighted estimates of second order using the scaling operator

$$S = t\partial_t + x \cdot \nabla_x.$$

As a consequence we get the following decay estimate $\|u_t\|_{Lm+1} \le Ct^{-1/(m+1)}$, see Proposition 3.4. This holds in all dimensions $n \ge 1$. We can estimate $\|u_t\|_{L2}$ by the finite propagation speed and Hölder inequality. However, we obtain a faster decay rate if we replace the finite propagation speed with the estimate of $E_{\text{ext}}(t)$ in Proposition 2.1. The strongest result is given below.

Proposition 3.1. *Let u be a solution of equation (1.5) with compactly supported initial data $(u_0, u_1) \in H^2 \times H^1$. For any $n \geq 1$,*

$$\|u_t\|_{L^2} \leq Ct^{\frac{(1+\delta)n}{4}\frac{m-1}{m+1}-\frac{1}{m+1}} \ln^{\frac{1}{2}} t, \quad t \geq 2,$$

where $\delta = \frac{(m-1)(m+n+3)}{(m+1)^2-(m-1)n}$. (Notice that $\delta < 1$ if $m < 1 + \frac{2}{n}$.)

The proof combines two simple lemmas. We consider the energy functional

$$\mathcal{E}(u,t) = \frac{1}{2}\int (u_t^2 + |\nabla u|^2)dx$$

for various functions u.

Lemma 3.2. *Let u be a solution of equation (1.5) with compactly supported initial data $(u_0, u_1) \in H^2 \times H^1$. Then*

$$\mathcal{E}(Su,t) \leq 2\mathcal{E}(Su + \frac{2-m}{m-1}u, 0) + 2\left(\frac{2-m}{m-1}\right)^2 \mathcal{E}(u,0),$$

where the exponent $m > 1$. Hence $\mathcal{E}(Su,t) \leq C$ for all $t \geq 0$.

Proof. Consider the family of scaled functions

$$u_\lambda(t,x) = e^{\frac{2-m}{m-1}\lambda}u(e^\lambda t, e^\lambda x), \quad \lambda \in \mathbb{R},$$

and notice that

$$\frac{d}{d\lambda}u_\lambda(t,x)|_{\lambda=0} = Su + \frac{2-m}{m-1}u.$$

It is easy to verify that u and u_λ satisfy the same equation (1.5), i.e.,

$$(\partial_t^2 - \Delta)u + |\partial_t u|^{m-1}\partial_t u = 0$$

and

$$(\partial_t^2 - \Delta)u_\lambda + |\partial_t u_\lambda|^{m-1}\partial_t u_\lambda = 0.$$

Subtracting these, we have the equation

$$(\partial_t^2 - \Delta)(u_\lambda - u) + |\partial_t u_\lambda|^{m-1}\partial_t u_\lambda - |\partial_t u|^{m-1}\partial_t u = 0.$$

We multiply with $\partial_t(u_\lambda - u)$ and use the monotonicity of the damping, namely

$$(|\partial_t u_\lambda|^{m-1}\partial_t u_\lambda - |\partial_t u|^{m-1}\partial_t u)(\partial_t u_\lambda - \partial_t u) \geq 0.$$

Thus we obtain the inequality

$$(\partial_t^2 - \Delta)(u_\lambda - u) \cdot \partial_t(u_\lambda - u) \leq 0$$

and, after integration over $\mathbb{R}^n$, the estimate

$$\frac{d}{dt}\mathcal{E}(u_\lambda - u, t) \leq 0.$$

Hence,

$$\mathcal{E}(u_\lambda - u, t) \leq \mathcal{E}(u_\lambda - u, 0).$$

We divide by λ^2 and pass to the limit as $\lambda \to 0$:

$$\mathcal{E}(Su + \frac{2-m}{m-1}u, t) \leq \mathcal{E}(Su + \frac{2-m}{m-1}u, 0).$$

Using $\mathcal{E}(u+v,t) \leq 2\mathcal{E}(u,t) + 2\mathcal{E}(v,t)$ and $\mathcal{E}(cu,t) = c^2\mathcal{E}(u,t)$, we complete the proof of Lemma 3.2. □

In the next lemma we express u_t in terms of Su from the identities

$$tu_t = Su - x\cdot\nabla u, \tag{3.1}$$

$$tu_{tt} = (Su - x\cdot\nabla u)_t - u_t. \tag{3.2}$$

Lemma 3.3. *The following identities hold for $t>0$:*

$$(i) \quad \int u_{tt}u_t\ dx = \frac{n-2}{2t}\int u_t^2\ dx + \frac{1}{t}\int (Su)_t u_t\ dx,$$

$$(ii) \quad \int -\Delta u u_t\ dx = \frac{n-2}{2t}\int |\nabla u|^2\ dx + \frac{1}{t}\int \nabla(Su)\cdot\nabla u\ dx.$$

Proof. To show (i), we apply (3.2). We can write

$$\begin{aligned} tu_{tt}u_t &= (Su)_t u_t - (x\cdot\nabla u_t)u_t - u_t^2 \\ &= (Su)_t u_t - \frac{1}{2}x\cdot\nabla u_t^2 - u_t^2 \\ &= (Su)_t u_t - \frac{1}{2}\mathrm{div}(xu_t^2) + \left(\frac{n}{2}-1\right)u_t^2. \end{aligned}$$

Integrating on $\mathbb{R}^n$ and using the compact support of $u(t,\cdot)$, we derive (i). The proof of (ii) is similar. □

To apply Lemma 3.3, we multiply equation (1.5) by u_t and integrate on $\mathbb{R}^n$:

$$\int |u_t|^{m+1}dx = -\int u_{tt}u_t + \int \Delta u u_t dx.$$

It follows from Lemma 3.3 that

$$\begin{aligned} \int |u_t|^{m+1}dx &= \frac{2-n}{2t}\int u_t^2 dx + \frac{2-n}{2t}\int |\nabla u|^2 dx \\ &\quad -\frac{1}{t}\int (Su)_t u_t dx - \frac{1}{t}\int \nabla(Su)\cdot\nabla u dx \\ &\leq \frac{2-n}{t}\mathcal{E}(u,t) + \frac{C}{t}\mathcal{E}^{1/2}(u,t)\mathcal{E}^{1/2}(Su,t). \end{aligned}$$

From this inequality and Lemma 3.2 we deduce the following decay estimate.

Proposition 3.4. *Let u be a solution of (1.5) with compactly supported initial data $(u_0,u_1)\in H^2\times H^1$. Then*

$$\|u_t\|_{L^{m+1}} \leq Ct^{-\frac{1}{m+1}}, \quad t\geq 2,$$

for all dimensions $n\geq 1$ and exponents $m>1$.

We can now bound $\|u_t\|_{L^2}$ by Hölder's inequality and the finite propagation speed of equation (1.5).

Corollary 3.5. *Under the assumptions of Proposition 3.4,*

$$\|u_t\|_{L^2} \leq Ct^{\frac{n}{2}\frac{m-1}{m+1}-\frac{1}{m+1}}, \quad t\geq 2.$$

In particular, $\|u_t\|_{L^2}\to 0$ as $t\to\infty$ if $1<m<1+\frac{2}{n}$ holds.

Proof. We use Proposition 3.4 and the inequality

$$\|u_t\|_{L^2} \leq [\mathrm{vol}(\mathbf{B}^n)(t+R)^n]^{\frac{1}{2}\frac{m-1}{m+1}} \|u_t\|_{L^{m+1}}.$$

□

Finally, we derive a stronger decay estimate for $\|u_t\|_{L^2}$ combining Proposition 3.4 and Proposition 2.1.

Proof of Proposition 3.1. We split the norm $\|u_t\|_{L^2}$ over interior and exterior regions:

$$\|u_t\|_{L^2} \leq \|u_t\|_{L^2(|x|\leq t^{(1+\delta)/2})} + \|u_t\|_{L^2(|x|>t^{(1+\delta)/2})}.$$

The first term is bounded by Hölder's inequality and Proposition 3.4, while the second term is bounded by Proposition 2.1:

$$\begin{aligned} \|u_t\|_{L^2} &\leq [\mathrm{vol}(\mathbf{B}^n)(t+R)^{\frac{1+\delta}{2}n})]^{\frac{1}{2}\frac{m-1}{m+1}} \|u_t\|_{L^{m+1}(|x|<t^{\frac{1+\delta}{2}})} + 2^{\frac{1}{2}} E_{\mathrm{ext}}^{\frac{1}{2}}(u,t) \\ &\leq Ct^{\frac{(1+\delta)n}{4}\frac{m-1}{m+1}-\frac{1}{m+1}} + Ct^{\frac{1}{2}[\frac{n}{2}+1-\delta(\frac{m+1}{m-1}-\frac{n}{2})]} \ln^{\frac{1}{2}} t. \end{aligned}$$

The optimal δ is such that the two powers of t are equal. Thus,

$$\delta = \frac{(m-1)(m+n+3)}{(m+1)^2-(m-1)n}.$$

It is easy to check that $\delta < 1$ when $1 < m < 1 + \frac{2}{n}$, so the estimate here is stronger than the estimate in Corollary 3.5. □

4. L^p Estimates for ∇u

We establish an L^p estimate of ∇u in terms of $\partial_t u$ for solutions of the wave equation with nonlinear damping

$$\partial_t^2 u - \Delta u = -|\partial_t u|^{m-1}\partial_t u. \tag{4.1}$$

Possible applications include the asymptotic behavior of energy as well as the regularizing effect of nonlinear damping on ∇u. Here we consider the former question only. For our goal the exponents $p \leq 2$ are entirely sufficient, although the argument can be modified to cover all exponents $2 < p \leq m+1$.

Proposition 4.1. *Assume that $n \geq 3$. Let u be a solution of (4.1) with data*

$$u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1.$$

There exist constants B_p, $C_{p,d}$, and $D_{p,d}$, such that

$$\begin{aligned} &\left(\int_0^t \int (s+|x|^2)^{d/2} |\nabla u|^p dx ds\right)^{1/p} \\ \leq\ & B_p \left(\int_0^t \int |x|^{d+p} |\partial_s u|^{mp} dx ds\right)^{1/p} \\ &+ C_{p,d} \left(\int_0^t \int s^{d/2} |\partial_s u|^p dx ds\right)^{1/p} \\ &+ D_{p,d} \left(\sum_{k=0,1} (\|(\sqrt{-\Delta})^k u_0\|_{L^1} + \|(\sqrt{-\Delta})^k u_0\|_{L^p}) + \|u_1\|_{L^1} + \|u_1\|_{L^p}\right), \end{aligned}$$

for any p and d satisfying

$$\frac{n}{n-1} \leq p \leq 2 \quad and \quad -2 < d \leq 0.$$

Here every constant depends only on its subscripts p, or p and d.

The proof of Proposition 4.1 is based on two weighted L^p estimates for linear equations. First we consider the wave equation

$$\partial_t^2 u - \Delta u = f \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^n, \tag{4.2}$$

which we rewrite in the form

$$(\partial_t + \sqrt{-\Delta})^2 u = f + 2\sqrt{-\Delta}\ \partial_t u \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^n. \tag{4.3}$$

This is a parabolic equation for u with a new source depending on f and $\partial_t u$. We have an estimate of ∇u by the two terms in the source.

Lemma 4.2. *Assume that* $n \geq 3$. *Let* u *be a solution of equation (4.2) with zero initial data* $u|_{t=0} = \partial_t u|_{t=0} = 0$. *There exist constants* C_p *and* $C_{p,d}$, *such that* u *satisfies*

$$\begin{aligned} &\left(\int_0^t \int (s + |x|^2)^{d/2} |\nabla u|^p dx ds\right)^{1/p} \\ \leq\ & C_p \left(\int_0^t \int |x|^d |(\sqrt{-\Delta})^{-1} f|^p dx ds\right)^{1/p} \\ &+ C_{p,d} \left(\int_0^t \int s^{d/2} |\partial_s u|^p dx ds\right)^{1/p}, \end{aligned}$$

whenever p and d satisfy $1 < p < \infty$ *and* $-2 < d \leq 0$.

The above result will be applied to the wave equation (4.1) with the damping treated as a source. To insure zero initial data in Lemma 4.2, we subtract the solution u_l of the linear parabolic equation

$$(\partial_t + \sqrt{-\Delta})^2 u_l = 0 \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^n. \tag{4.4}$$

Formula (4.3) explains why u_l is more convenient than a solution of the wave equation with identical initial data.

Lemma 4.3. *Assume that* $n \geq 3$. *Let* u_l *be a solution of equation (4.4) with initial data* $u_l|_{t=0} = u_0$ *and* $\partial_t u_l|_{t=0} = u_1$. *There exists a constant* $D_{p,d}$, *such that* u_l *satisfies*

$$\begin{aligned} &\left(\int_0^t \int (s + |x|^2)^{d/2} |\nabla u_l|^p dx ds\right)^{1/p} \\ \leq\ & D_{p,d} \left(\sum_{k=0,1} (\|(\sqrt{-\Delta})^k u_0\|_{L^1} + \|(\sqrt{-\Delta})^k u_0\|_{L^p}) + \|u_1\|_{L^1} + \|u_1\|_{L^p}\right) \end{aligned}$$

whenever p and d satisfy $n/(n-1) \leq p \leq 2$ *and* $-2 < d \leq 0$.

To show Lemma 4.2 and Lemma 4.3 we use some basic facts from harmonic analysis presented in Appendix A. In particular, we rely on L^p estimates for certain convolution operators in weighted spaces satisfying the A_p condition. Proposition 4.1 readily follows from the two lemmas above.

Proof of Proposition 4.1. We rewrite equation (4.1) as

$$(\partial_t + \sqrt{-\Delta})^2 u = -|\partial_t u|^{m-1}\partial_t u + 2\sqrt{-\Delta}\ \partial_t u.$$

Applying Lemma 4.2 and Lemma 4.3, we have

$$\begin{aligned}
&\left(\int_0^t \int (s+|x|^2)^{d/2}|\nabla u|^p dxds\right)^{1/p} \qquad (4.5)\\
\le\ & C_p\left(\int_0^t \int |x|^d|(\sqrt{-\Delta})^{-1}|\partial_s u|^{m-1}\partial_s u|^p dxds\right)^{1/p}\\
&+C_{p,d}\left(\int_0^t \int s^{d/2}|\partial_s u|^p dxds\right)^{1/p}\\
&+D_{p,d}\left(\sum_{k=0,1}(\|(\sqrt{-\Delta})^k u_0\|_{L^1} + \|(\sqrt{-\Delta})^k u_0\|_{L^p}) + \|u_1\|_{L^1} + \|u_1\|_{L^p}\right)
\end{aligned}$$

with p and d satisfying the conditions there. To complete the proof we simplify

$$C_p\int_0^t \int |x|^d|(\sqrt{-\Delta})^{-1}|\partial_s u|^{m-1}\partial_s u|^p dxds,$$

so that the constant remains independent of d. First we use Hardy's inequality

$$\int |x|^d|f(x)|^p dx \le \left(\frac{p}{n+d}\right)^p \int |x|^{d+p}|\nabla f(x)|^p dx$$

with $f = (\sqrt{-\Delta})^{-1}|\partial_s u|^{m-1}\partial_s u$, $0 \le s \le t$, and $d > -n$:

$$\begin{aligned}
&\left(\int_0^t \int |x|^d|(\sqrt{-\Delta})^{-1}|\partial_s u|^{m-1}\partial_s u|^p dxds\right)^{1/p} \qquad (4.6)\\
&\le \frac{p}{n+d}\left(\int_0^t \int |x|^{d+p}|\nabla(\sqrt{-\Delta})^{-1}|\partial_s u|^{m-1}\partial_s u|^p dxds\right)^{1/p}.
\end{aligned}$$

Next we need weighted L^p estimates for the Riesz transform

$$Rf = \nabla(\sqrt{-\Delta})^{-1}f \quad \text{or} \quad Rf(x) = (2\pi)^{-n}\int \frac{i\xi}{|\xi|}e^{i\xi\cdot x}\hat{f}(\xi)d\xi, \quad x\in\mathbb{R}^n.$$

It is well known that R is a convolution transform:

$$Rf = K_R * f \quad \text{with} \quad K_R(x) = r_n\frac{x}{|x|^{n+1}}, \quad r_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}.$$

Moreover K_R satisfies the conditions of Lemma A.1. Since the weight

$$w_1^{d+p}(x) = |x|^{d+p}, \quad -n < d+p < n(p-1),$$

satisfies the $A_p(\mathbb{R}^n)$ condition, according to Lemma A.2, R is a bounded operator in the space $L^p(w_1^{d+p})$:

$$\|\nabla(\sqrt{-\Delta})^{-1}f\|_{L^p(w_1^{d+p})} \le K_p(w_1^{d+p})\|f\|_{L^p(w_1^{d+p})}.$$

The range for d is $-n-p < d < n(p-1)-p$, so it includes the range $-2 \le d \le 0$ under the condition $0 \le n(p-1)-p$, or

$$p \ge \frac{n}{n-1}. \qquad (4.7)$$

We use a standard convexity argument to find a uniform constant K_p depending on the constants $K_p(w_1^{-2})$ and $K_p(w_1^0)$:

$$\|\nabla(\sqrt{-\Delta})^{-1} f\|_{L^p(w_1^{d+p})} \leq K_p \|f\|_{L^p(w_1^{d+p})}, \quad -2 \leq d \leq 0.$$

Combining this inequality with Hardy's inequality (4.6), we have

$$\begin{aligned}&\left(\int_0^t \int |x|^d |(\sqrt{-\Delta})^{-1} |\partial_s u|^{m-1} \partial_s u|^p dx ds\right)^{1/p} \\ &\leq \frac{pK_p}{n+d} \left(\int_0^t \int |x|^{d+p} |\partial_s u|^{mp} dx ds\right)^{1/p},\end{aligned}$$

whenever the conditions $-2 \leq d \leq 0$ and (4.7) hold. A substitution into inequality (4.5) yields the final estimate

$$\begin{aligned}&\left(\int_0^t \int (s+|x|^2)^{d/2} |\nabla u|^p dx ds\right)^{1/p} \\ \leq\ & B_p \left(\int_0^t \int |x|^{d+p} |\partial_s u|^{mp} dx ds\right)^{1/p} \\ &+C_{p,d} \left(\int_0^t \int s^{d/2} |\partial_s u|^p dx ds\right)^{1/p} \\ &+D_{p,d} \left(\sum_{k=0,1} (\|(\sqrt{-\Delta})^k u_0\|_{L^1} + \|(\sqrt{-\Delta})^k u_0\|_{L^p}) + \|u_1\|_{L^1} + \|u_1\|_{L^p}\right)\end{aligned}$$

with $B_p = pC_pK_p/(n-2)$. The proof of Proposition 4.1 is complete. □

We will verify the two lemmas used in the proof of Proposition 4.1.

Proof of Lemma 4.2. The first step is to derive a suitable representation for ∇u. Solving equation (4.3) for u with zero initial data, we obtain

$$u(t) = \int_0^t (t-s) e^{-(t-s)\sqrt{-\Delta}} f(s) ds + 2\int_0^t (t-s) e^{-(t-s)\sqrt{-\Delta}} \sqrt{-\Delta} \partial_s u ds.$$

Let us apply ∇ to both sides of the equality and use $e^{-t\sqrt{-\Delta}}\sqrt{-\Delta} = -\partial_t e^{-t\sqrt{-\Delta}}$.

$$\begin{aligned}\nabla u(t) &= -\int_0^t (t-s)\partial_t \nabla e^{-(t-s)\sqrt{-\Delta}} (\sqrt{-\Delta})^{-1} f(s) ds \\ &\quad -2\int_0^t (t-s)\partial_t \nabla e^{-(t-s)\sqrt{-\Delta}} \partial_s u ds.\end{aligned}$$

Introducing the operator

$$Tf(t,x) = -\int_0^t (t-s)\partial_t \nabla e^{-(t-s)\sqrt{-\Delta}} f(s,x) ds, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

we can rewrite ∇u in the form

$$\nabla u = T(\sqrt{-\Delta})^{-1} f + 2T\partial_t u. \tag{4.8}$$

The next step is to deduce weighted L^p estimates for T. It is easy to see that T is a convolution operator:

$$Tf = K_T * f \quad \text{with} \quad K_T(t,x) = -H(t) t \partial_t \nabla P_t(x),$$

where H is the Heaviside function

$$H(t) = \begin{cases} 0, & \text{if} \quad t < 0, \\ 1, & \text{if} \quad t \geq 0, \end{cases}$$

and $P_t(x)$ is the Poisson kernel

$$P_t(x) = (2\pi)^{-n} \int e^{-t|\xi| + i\xi\cdot x} d\xi.$$

We can compute the Fourier transform $\hat{K}_T(\tau, \xi)$ using the partial Fourier transform $\hat{P}_t(\xi) = e^{-t|\xi|}$ and identity $K_T(t,x) = -H(t)t\partial_t \nabla P_t(x)$:

$$\begin{aligned} \hat{K}_T(\tau,\xi) &= -i\xi \int_0^\infty e^{-it\tau} t\partial_t e^{-t|\xi|} dt \\ &= \frac{i|\xi|\xi}{(|\xi| + i\tau)^2}. \end{aligned}$$

Thus we obtain

$$|\hat{K}(\tau,\xi)| \leq C, \quad (\tau,\xi) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\}. \tag{4.9}$$

The Poisson kernel $P_t(x)$ is given explicitly by

$$P_t(x) = p_n \frac{t}{(t^2 + x^2)^{(n+1)/2}}, \quad p_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}. \tag{4.10}$$

Differentiating $K_T(t,x) = -H(t)t\partial_t \nabla P_t(x)$ once, we have

$$|(\partial_t, \nabla)K(t,x)| \leq \frac{C}{(t^2 + x^2)^{(n+2)/2}}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \setminus \{0\}. \tag{4.11}$$

It follows from (4.9) and (4.11) that K_T meets the conditions of Lemma A.1. Hence T is a bounded operator in L^p for $1 < p < \infty$. Such operators are also bounded in $L^p(w)$ for any weight w satisfying the A_p condition in $\mathbb{R}_+ \times \mathbb{R}^n$. Lemma A.2 shows that

$$\begin{aligned} w_1^d(t,x) &= |x|^d, \quad -n < d < n(p-1), \\ w_2^d(t,x) &= t^{d/2}, \quad -2 < d < 2(p-1), \end{aligned}$$

are A_p weights, so T is a bounded operator in $L^p(w_k^d)$ for $k = 1, 2$. Thus

$$\|Tf\|_{L^p(w_k^d)} \leq C_p(w_k^d)\|f\|_{L^p(w_k^d)}, \quad k = 1, 2.$$

It remains to determine how the constant $C_p(w_1^d)$ depends on d, which is important for our applications of this estimate. A standard convexity argument implies that $C_p(w_1^d)$ can be bounded in terms of $C_p(w_1^{-2})$ and $C_p(w_1^0)$ for $d \in [-2, 0]$. Let us denote the upper bound by C_p. Thus

$$\|Tf\|_{L^p(w_1^d)} \leq C_p\|f\|_{L^p(w_1^d)}, \quad -2 \leq d \leq 0.$$

Moreover, the weighted estimates in $L^p(w_1^d)$ and $L^p(w_2^d)$ admit restrictions to each finite interval $[0,t] \subset \mathbf{R}_+$:

$$\begin{aligned} &\left(\int_0^t \int w_k^d(s,x)|Tf(s,x)|^p dx ds \right)^{1/p} \\ \leq \ & C_{p,d}^{(k)} \left(\int_0^t \int w_k^d(s,x)|f(s,x)|^p dx ds \right)^{1/p}, \quad k = 1, 2. \end{aligned} \tag{4.12}$$

Here $C^{(1)}_{p,d} = C_p$ and $C^{(2)}_{p,d} = C_p(w^d_2)$. (We will write $C_{p,d}$ for the latter constant.) Both estimates are valid if $-2 < d \le 0$. To complete the proof, we notice that

$$(s + |x|^2)^{d/2} \le |x|^d \quad \text{and} \quad (s + |x|^2)^{d/2} \le s^{d/2},$$

and apply estimates (4.12) of T to the representation (4.8) of ∇u. □

Proof of Lemma 4.3. The solution of equation (4.4) is given by

$$u_l(t, x) = P_t(x) * u_0(x) + tP_t(x) * (u_1(x) + \sqrt{-\Delta} u_0(x)).$$

To estimate the two convolutions, we apply three inequalities for $P_t(x)$ which are verified directly from formula (4.10).

If $0 < t \le 1$,

$$\|\nabla P_t * f\|_{L^p} \le C\|\nabla f\|_{L^p} \quad \text{and} \quad \|\nabla P_t * f\|_{L^p} \le \frac{C}{t}\|f\|_{L^p}. \tag{4.13}$$

If $t > 1$,

$$\|\nabla P_t * f\|_{L^p} \le \frac{C}{t^{n(p-1)/p+1}}\|f\|_{L^1}. \tag{4.14}$$

We show only the estimate of $\nabla tP_t(x) * (u_1(x) + \sqrt{-\Delta} u_0(x))$, since the other estimate is similar. Let $f = u_1(x) + \sqrt{-\Delta} u_0(x)$. By the second inequality in (4.13),

$$\int_0^1 \|t\nabla P_t * f\|^p_{L^p} t^{d/2} dt \le C\|f\|^p_{L^p} \int_0^1 t^{d/2} dt.$$

Recall that $d > -2$, so the integral converges. Thus

$$\int_0^1 \|t\nabla P_t * f\|^p_{L^p} t^{d/2} dt \le C(\|u_1\|^p_{L^p} + \|\sqrt{-\Delta} u_0\|^p_{L^p}).$$

We apply (4.14) to bound

$$\int_1^\infty \|t\nabla P_t * f\|^p_{L^p} t^{d/2} dt \le C\|f\|^p_{L^1} \int_1^\infty t^{d/2-n(p-1)} dt.$$

Notice that $d/2 - n(p-1) \le -n/(n-1)$ for $p \ge n/(n-1)$. Hence the integral converges:

$$\int_1^\infty \|t\nabla P_t * f\|^p_{L^p} t^{d/2} dt \le C(\|u_1\|^p_{L^1} + \|\sqrt{-\Delta} u_0\|^p_{L^1}).$$

Adding the estimates for $(0, 1]$ and $[1, \infty)$, we obtain

$$\int_0^\infty \|t\nabla P_t * f\|^p_{L^p} t^{d/2} dt \le C(\|u_1\|^p_{L^p} + \|\sqrt{-\Delta} u_0\|^p_{L^p} + \|u_1\|^p_{L^1} + \|\sqrt{-\Delta} u_0\|^p_{L^1}),$$

for $p \ge n/(n-1)$. The proof is complete for this term.

There is a similar estimate for the other term $\nabla P_t(x) * u_0(x)$. The main difference is that we apply the first inequality in (4.13) to the integral on $(0, 1]$. □

We conclude this section with a corollary of Proposition 4.1 relating the L^{m+1} norm of $\partial_t u$ and the $L^{(m+1)/m}$ norm of ∇u. Recall that the former can be bounded by the damping in equation (4.1). We choose $p = (m+1)/m$ and $d = a-(m+1)/m$ in Proposition 4.1. Here $a \in (0,1)$. The corresponding estimate is

$$\begin{aligned}
(4.15) \qquad & \left(\int_0^t \int (s+|x|^2)^{\frac{a}{2}-\frac{m+1}{2m}} |\nabla u|^{\frac{m+1}{m}} dxds\right)^{\frac{m}{m+1}} \\
\leq \quad & B_{(m+1)/m} \left(\int_0^t \int |x|^a |\partial_s u|^{m+1} dxds\right)^{\frac{m}{m+1}} \\
& + C_{(m+1)/m, a-(m+1)/m} \left(\int_0^t \int s^{\frac{a}{2}-\frac{m+1}{2m}} |\partial_s u|^{\frac{m+1}{m}} dxds\right)^{\frac{m}{m+1}} + C_0,
\end{aligned}$$

where C_0 is a constant depending on m, d, and the initial data. We need an upper bound on the $L^{(m+1)/m}$ norm of $\partial_t u$, i.e., the integral

$$I_2 = \int_0^t \int s^{\frac{a}{2}-\frac{m+1}{2m}} |\partial_s u|^{\frac{m+1}{m}} dxds.$$

We will consider separately the cases of small and large t.

If $t \leq 1$, we have

$$I_2 \leq \sup_{0\leq s\leq 1} \|\partial_s u\|_{L^{(m+1)/m}}^{(m+1)/m} \int_0^t s^{\frac{a}{2}-\frac{m+1}{2m}} ds.$$

Since $(m+1)/m \leq 2$ and $u(t,\cdot)$ is compactly supported, we obtain

$$I_2 \leq C(1+E(0)), \quad t \leq 1.$$

If $t > 1$, the finite propagation speed and Hölder's inequality yield

$$\begin{aligned}
(4.16) \qquad & \int_1^t \int s^{\frac{a}{2}-\frac{m+1}{2m}} |\partial_s u|^{\frac{m+1}{m}} dxds \\
\leq \quad & \left(\int_1^t \int_{|x|\leq s+R} s^{\frac{a}{2}-\frac{m+1}{2(m-1)}} dxds\right)^{\frac{m-1}{m}} \left(\int_1^t \int s^{\frac{a}{2}} |\partial_s u|^{m+1} dxds\right)^{\frac{1}{m}}.
\end{aligned}$$

From vol $B(s+R) \leq C(s+R)^n$, the first factor is bounded by

$$\int_1^t s^{n+\frac{a}{2}-\frac{m+1}{2(m-1)}} ds \leq K,$$

if $n + a/2 - (m+1)/(2(m-1)) < -1$. Applying Young's inequality,

$$\begin{aligned}
\int_1^t \int s^{\frac{a}{2}-\frac{m+1}{2m}} |\partial_s u|^{\frac{m+1}{m}} dxds \quad & \leq \quad K^{\frac{m-1}{m}} \left(\int_1^t \int s^{\frac{a}{2}} |\partial_s u|^{m+1} dxds\right)^{\frac{1}{m}} \\
& \leq \quad C_\epsilon + \epsilon \int_0^t \int s^{\frac{a}{2}} |\partial_s u|^{m+1} dxds,
\end{aligned}$$

for any $\epsilon > 0$. Thus

$$I_2 \leq C_\epsilon(1+E(0)) + \epsilon \int_0^t \int s^{\frac{a}{2}} |\partial_s u|^{m+1} dxds, \quad t > 1.$$

The estimates for I_2 together with estimate (4.15) yield the following result.

Corollary 4.4. *Assume that $n \geq 3$. Let u be a solution of (4.1) with initial data*

$$u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1.$$

There exist constants F_m and $G_{m,a}$, such that

$$\begin{aligned}&\int_0^t\int(s+|x|^2)^{\frac{a}{2}-\frac{m+1}{2m}}|\nabla u|^{\frac{m+1}{m}}dxds\\ \le\ & F_m\int_0^t\int(s+|x|^2)^{\frac{a}{2}}|\partial_s u|^{m+1}dxds+G_{m,a},\end{aligned}$$

for m and a satisfying

$$n+\frac{a}{2}-\frac{m+1}{2(m-1)}<-1 \quad \text{and} \quad 0<a<1.$$

The constant $G_{m,a}$ depends also on u_0 and u_1.

It is sufficient to choose $F_m=(2B_{(m+1)/m})^{(m+1)/m}$, which is still a constant independent of a. We will determine the optimal a in Section 5. The above condition on m is equivalent to

$$m<1+\frac{1}{n+1/2+a/2}.$$

Thus $m\le 1+1/(n+1)$ will be a stronger condition independent of a. However, such a restriction does not seem optimal. We can do better if we use the suppressed support instead of finite propagation speed in (4.16). The resulting sharper estimate of the first integral will imply a weaker restriction on m. We do not pursue this estimate since it is unlikely to give the optimal condition $m<1+2/n$.

5. Decay Rate of $\|\nabla u\|_{L^2}$

We prove that the energy of equation (1.5) decays polynomially as $t\to\infty$.

Proof of Theorem 1.1. We can assume that $t\ge 2$. The weighted energy identity is

$$\frac{d}{dt}\int\frac{w}{2}(u_t^2+|\nabla u|^2)dx=-\int w|u_t|^{m+1}dx+R_1+R_2, \tag{5.1}$$

where

$$\begin{aligned}R_1 &= \int\frac{w_t}{2}(u_t^2+|\nabla u|^2)dx,\\ R_2 &= \int\nabla w\cdot u_t\nabla u\ dx.\end{aligned}$$

Here the weight is $w(t,x)=(t+|x|^2)^{a/2}$ with a small constant $a\in(0,1)$. The exact conditions on a are given later. An important property of w is that

$$w_t=\frac{a}{2}(t+|x|^2)^{a/2-1},\quad \nabla w=ax(t+|x|^2)^{a/2-1},$$

so w_t is much smaller than $|\nabla w|$ when $|x|$ is large. More precisely, $|\nabla w|=2|x|w_t$.

Our goal is to show that the right side of identity (5.1) belongs to $L^1(\mathbb{R}_+)$ for sufficiently small a. The computations are elementary, based on Young's and Hölder's inequalities, with the exception of an $L^{(m+1)/m}$ estimate for ∇u established in Corollary 4.4. For convenience we restate this result.

Assume that m and a satisfy

$$n+\frac{a}{2}-\frac{m+1}{2(m-1)}<-1 \quad \text{and} \quad 0<a<1.$$

There exist constants F_m and $G_{m,a}$ such that the solution of equation (1.5) satisfies

$$\int_0^t \int (s+|x|^2)^{\frac{a}{2}-\frac{m+1}{2m}} |\nabla u|^{\frac{m+1}{m}} dxds \tag{5.2}$$
$$\leq F_m \int_0^t \int (s+|x|^2)^{\frac{a}{2}} |u_s|^{m+1} dxds + G_{m,a}.$$

The constant $G_{m,a}$ depends also on the initial data u_0 and u_1.

We begin with an upper bound of R_2. Applying Young's inequality, we obtain

$$\begin{aligned} |\nabla w \cdot u_t \nabla u| &\leq a|x|(t+|x|^2)^{\frac{a}{2}-1}|u_t||\nabla u| \\ &\leq \frac{a}{m+1}(t+|x|^2)^{\frac{a}{2}}|u_t|^{m+1} \\ &\quad + \frac{am}{m+1}|x|^{\frac{m+1}{m}}(t+|x|^2)^{\frac{a}{2}-\frac{m+1}{m}}|\nabla u|^{\frac{m+1}{m}}. \end{aligned}$$

Since $|x| \leq (t+|x|^2)^{1/2}$, we have the following estimate after integration:

$$\begin{aligned} R_2 &\leq \frac{a}{m+1}\int (t+|x|^2)^{\frac{a}{2}}|u_t|^{m+1}dx \\ &\quad + \frac{am}{m+1}\int (t+|x|^2)^{\frac{a}{2}-\frac{m+1}{2m}}|\nabla u|^{\frac{m+1}{m}}dx. \end{aligned} \tag{5.3}$$

The two terms will be bounded separately.

We proceed to derive a similar estimate for R_1. One part of R_1 is readily bounded by Young's inequality:

$$\begin{aligned} \frac{w_t}{2}u_t^2 &= \frac{a}{4}(t+|x|^2)^{\frac{a}{2}-1}|u_t|^2 \\ &\leq \frac{a}{4(m+1)}(t+|x|^2)^{\frac{a}{2}}|u_t|^{m+1} \\ &\quad + \frac{a(m-1)}{4(m+1)}(t+|x|^2)^{\frac{a}{2}-\frac{m+1}{m-1}}. \end{aligned}$$

Hence,

$$\begin{aligned} \int \frac{w_t}{2}u_t^2 dx &\leq \frac{a}{4(m+1)}\int (t+|x|^2)^{\frac{a}{2}}|u_t|^{m+1}dx \\ &\quad + \frac{a(m-1)}{4(m+1)}\int (t+|x|^2)^{\frac{a}{2}-\frac{m+1}{m-1}}dx. \end{aligned} \tag{5.4}$$

The remaining part of R_1, which involves $\frac{w_t}{2}|\nabla u|^2$, needs a different application Young's inequality:

$$\begin{aligned} \frac{w_t}{2}|\nabla u|^2 &= \frac{a}{4}(t+|x|^2)^{\frac{a}{2}-1}|\nabla u|^2 \\ &\leq \frac{am}{4(m+1)}(t+|x|^2)^{\frac{a}{2}-\frac{m+1}{2m}}|\nabla u|^{\frac{m+1}{m}} \\ &\quad + \frac{a}{4(m+1)}(t+|x|^2)^{\frac{a}{2}-\frac{m+1}{2}}|\nabla u|^{m+1}. \end{aligned}$$

Integrating the last estimate, we have

$$\int \frac{w_t}{2}|\nabla u|^2 dx \quad \leq \quad \frac{am}{4(m+1)}\int (t+|x|^2)^{\frac{a}{2}-\frac{m+1}{m}}|\nabla u|^{\frac{m+1}{m}}dx$$

$$+\frac{a}{4(m+1)}\int (t+|x|^2)^{\frac{a}{2}-\frac{m+1}{2}}|\nabla u|^{m+1}dx. \tag{5.5}$$

We can now substitute estimates (5.3), (5.4), and (5.5) into identity (5.1) and integrate the resulting estimate on $[2,t]$:

$$\frac{1}{2}\int (s+|x|^2)^{\frac{a}{2}}(|u_s|^2+|\nabla u|^2)dx\Big|_2^t$$

$$\leq \quad d_1(t)+d_2(t)+d_3\int_2^t\int (s+|x|^2)^{\frac{a}{2}-\frac{m+1}{2m}}|\nabla u|^{\frac{m+1}{m}}dxds$$

$$-d_4\int_2^t\int (s+|x|^2)^{\frac{a}{2}}|u_s|^{m+1}dxds,$$

where the functions and constants are defined as follows:

$$d_1(t)=\frac{a(m-1)}{4(m+1)}\int_2^t\int (s+|x|^2)^{\frac{a}{2}-\frac{m+1}{m-1}}dxds,$$

$$d_2(t)=\frac{a}{4(m+1)}\int_2^t\int (s+|x|^2)^{\frac{a}{2}-\frac{m+1}{2}}|\nabla u|^{m+1}dxds,$$

$$d_3=\frac{5am}{4(m+1)}, \quad d_4=1-\frac{5a}{4(m+1)}.$$

Clearly $d_3\to 0$, while the other constant $d_4\to 1$ as $a\to 0$. Moreover, the functions $d_i(t)$, $i=1,2$, are bounded on $\mathbb{R}_+$. See Lemma 5.1 below. The main difficulty is to bound the third term by the fourth (damping) term. We apply estimate (5.2) and obtain

$$\frac{1}{2}\int (s+|x|^2)^{\frac{a}{2}}(|u_s|^2+|\nabla u|^2)dx\Big|_2^t$$

$$\leq \quad d_1(t)+d_2(t)+G_{m,a}$$

$$+(d_3F_m-d_4)\int_2^t\int (s+|x|^2)^{\frac{a}{2}}|u_s|^{m+1}dxds.$$

We can choose $a\in(0,1)$ sufficiently small to insure $d_3F_m-d_4<0$. Assuming that $d_1(t)$ and $d_2(t)$ are bounded on $\mathbb{R}_+$, we obtain

$$\frac{1}{2}\int (t+|x|^2)^{\frac{a}{2}}(|u_t|^2+|\nabla u|^2)dx\leq C.$$

Thus $E(t)\leq Ct^{-a/2}$. The proof is complete. □

It remains to verify the claim $d_i(t)\leq C$ for $i=1,2$.

Lemma 5.1. *Assume that* $m\leq 1+1/(n+1)$. *Then*

$$(i)\ d_1(t)\leq C, \quad (ii)\ d_2(t)\leq C, \quad t\geq 2.$$

Proof of Lemma 5.1. Part (i) follows if the exponent in $d_1(t)$ satisfies

$$\frac{a}{2} - \frac{m+1}{m-1} < -n-1.$$

This is equivalent to $m < 1 + 4/(2n+a)$, so it holds for $m \leq 1 + 1/(n+1)$.

To verify part (ii), we use the Gagliadro-Nirenberg inequality and the estimates $\|\nabla u\|_{L^2} \leq C$ and $\|\nabla^2 u\|_{L^2} \leq C$. Hence $\|\nabla u\|_{L^{m+1}} \leq C$, which implies

$$d_2(t) \leq C \int_2^\infty s^{\frac{a}{2} - \frac{m+1}{2}} ds.$$

The integral converges if $a/2-(m+1)/2 < -1$. This condition is met for sufficiently small a. We have completed the proof of Lemma 5.1. □

6. Open Problems

There are two types of open problems: relatively accessible and very difficult. Let us begin with the former type.

1. Generalize the polynomial decay in Theorem 1.1 for less regular initial data $(u_0, u_1) \in H^2 \times L^2$. Basically this means to prove Proposition 3.1 without using the scaling operator S and estimate more carefully $d_2(t)$ in Lemma 5.2.
2. Remove the requirement for compactly supported initial data.
3. Show the polynomial decay in Theorem 1.1 for all exponents $m < 1 + 2/n$.
4. We expect that $m = 1+2/n$ is the critical number for decay/non-decay of the energy. Namely, there exists a dense set of initial data for the nonlinear dissipative wave equation (1.5), such that $E(t)$ does not decay if $m > 1 + 2/n$.
5. The hard open question is to find the exact decay rate of the energy. Here we can use the regularizing effect of the nonlinear damping on ∇u. The scaling invariance of (1.5) with $m > 1$ may be crucial, as it helps transform local estimates into global ones.

Appendix A. Results from Harmonic Analysis

Let w be a non-negative locally integrable function on $\mathbb{R}^m$. The weighted space $L^p(w)$ consists of functions f whose p-th power is Lebesgue integrable with respect to the density w, i.e., the norm $\|f\|_{L^p(w)} < \infty$, where

$$\|f\|_{L^p(w)} = \left(\int |f(z)|^p w(z) dz \right)^{1/p}.$$

A weight w satisfies the A_p condition in $\mathbb{R}^m$, for $1 < p < \infty$, if there exists a constant C such that

$$\left(\frac{1}{|B|} \int_B w \right) \left(\frac{1}{|B|} \int_B w^{1-p'} \right)^{p-1} \leq C, \quad p' = p/(p-1), \tag{A.1}$$

for all balls $B \subset \mathbb{R}^m$. The A_1 condition is

$$\frac{1}{|B|} \int_B w \leq C w(z), \quad \text{a.e.} \quad z \in B, \tag{A.2}$$

for all balls $B \subset \mathbb{R}^m$.

The following is a classical result on Calderon-Zygmund singular operators, see Theorem 7.11 in [D].

Lemma A.1. *Let K be a tempered distribution in $\mathbb{R}^m$ which coincides with a locally integrable function in $\mathbb{R}^m \setminus \{0\}$. Assume that*

$$\begin{aligned} \hat{K}(\xi) &\leq C, \quad \xi \in \mathbb{R}^m, \\ |\nabla_z K(z)| &\leq \frac{C}{|z|^{m+1}}, \quad z \in \mathbb{R}^m \setminus \{0\}. \end{aligned}$$

For $1 < p < \infty$, the convolution $K * f$ satisfies

$$\|K * f\|_{L^p} \leq C_p \|f\|_{L^p}.$$

Moreover, for w satisfying the A_p condition (A.1),

$$\|K * f\|_{L^p(w)} \leq C_p(w) \|f\|_{L^p(w)}.$$

The next lemma verifies the A_p condition for two weights used in Section 4. This result is essentially know, but we give a short proof.

Lemma A.2. *Let (t, x) be the standard coordinates in $\mathbb{R}_+ \times \mathbb{R}^n$. For $1 < p < \infty$, the following hold:*
(i) $|x|^a \in A_p(\mathbb{R}_+ \times \mathbb{R}^n)$ *if* $-n < a < n(p-1)$,
(ii) $t^b \in A_p(\mathbb{R}_+ \times \mathbb{R}^n)$ *if* $-1 < b < p-1$.

To prove the above lemma, we need a basic factorization result about A_p weights:

$$w_1,\ w_2 \in A_1 \Rightarrow w_1 w_2^{1-p} \in A_p. \tag{A.3}$$

For the proof see Proposition 7.2 in [D].

Proof of Lemma A.2. We can write

$$a = a_1 + (1-p)a_2 \quad \text{with} \quad -n < a_1 \leq 0, \quad i = 1, 2.$$

By property (A.3) with $w_i = |x|^{a_i}$, claim (i) follows from

$$|x|^{a_i} \in A_1(\mathbb{R}_+ \times \mathbb{R}^n) \quad \text{for} \quad -n < a_1 \leq 0, \quad i = 1, 2.$$

Clearly it is sufficient to show that $|x|^{a_1}$ is an A_1 weight. Consider a ball B of radius r_0 centered at (t_0, x_0). The inequality to verify is

$$\frac{1}{|B|} \int_B |x|^{a_1} dx dt \leq C |y|^{a_1}, \quad \text{a.e.} \quad (s, y) \in B,$$

where C is independent of B and (s, y). Since $(t_0 - s)^2 + (x_0 - y)^2 \leq r_0^2$ and $a_1 \leq 0$, the strongest inequality corresponds to $s = t_0$ and $|y| = |x_0| + r_0$:

$$\frac{1}{r_0^{n+1}} \int_{(t_0 - t)^2 + (x_0 - x)^2 \leq r_0^2} |x|^{a_1} dx dt \leq C(|x_0| + r_0)^{a_1}.$$

Notice that the range of t in included in the interval $[t_0 - r_0, t_0 + r_0]$. Thus

$$\frac{1}{r_0^n} \int_{|x_0 - x| \leq r_0} |x|^{a_1} dx \leq C(|x_0| + r_0)^{a_1},$$

for all x_0 and r_0, will yield statement (i). We rewrite this inequality as

$$\frac{1}{r_0^n (|x_0| + r_0)^{a_1}} \int_{|x_0 - x| \leq r_0} |x|^{a_1} dx \leq C$$

and consider two cases for x_0 and r_0.

If $|x_0| \geq 2r_0$, then $|x_0 - x| \leq r_0$ implies $|x| \geq \frac{1}{2}|x_0|$. Hence

$$\frac{1}{r_0^n(|x_0|+r_0)^{a_1}} \int_{|x_0-x|\leq r_0} |x|^{a_1} dx \leq \frac{\left(\frac{1}{2}|x_0|\right)^{a_1} \operatorname{vol}(\{x : |x_0 - x| \leq r_0\})}{r_0^n(|x_0|+r_0)^{a_1}},$$

which is bounded by $\operatorname{vol}(\mathbf{B}^n)/2^{a_1}$, i.e., a constant depending only on a_1 and n.

If $|x_0| \leq 2r_0$, then $|x_0 - x| \leq r_0$ yields $|x| \leq 3r_0$. Thus

$$\frac{1}{r_0^n(|x_0|+r_0)^{a_1}} \int_{|x_0-x|\leq r_0} |x|^{a_1} dx \leq \frac{1}{r_0^n(|x_0|+r_0)^{a_1}} \int_{|x|\leq 3r_0} |x|^{a_1} dx.$$

The right side is dominated by $\operatorname{area}(\mathbf{S}^{n-1})3^{a_1+n}/(a_1+n)$. This completes the proof of claim (i). Similarly, we can verify claim (ii). □

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