Department of Mathematics
The University of Tennessee

Spring 2016
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Math 231:Introduction to Ordinary Differential Equations
Exam #1 (February 10, 2016)

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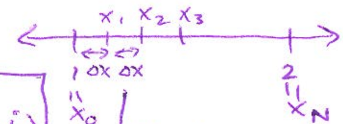
Please read each problem carefully, and indicate answers as directed. Your solutions **must** be supported by calculations or explanations: *no points will be given for answers that are not accompanied by, or are not consistent with, supporting work. Partial credit can be earned for steps that make reasonable progress towards a solution.*

Problem #	Points	Score
1	30	
2	30	
3	10	
4	30	
Total	100	

1. For every part of this problem, let $f(x) = 2x - 1$. We will find a formula for R_N for f on $[1, 2]$, and then compute $\int_1^2 f(x)\,dx$ by taking the limit $\lim_{N\to\infty} R_N$.

(a) [10 points] Give a formula for R_N, expressed using summation notation. Make clear in your work what Δx is equal to.

$$\Delta x = \frac{2-1}{N} = \frac{1}{N}$$



$x_i = 1 + i\left(\frac{1}{N}\right)$

$$R_N = \frac{1}{N}f(x_1) + \frac{1}{N}f(x_2) + \cdots + \frac{1}{N}f(x_N) = \boxed{\sum_{i=1}^{N} \frac{1}{N} f\left(1+\frac{i}{N}\right)}$$

$$= \frac{1}{N}f\left(1+\frac{1}{N}\right) + \frac{1}{N}f\left(1+2\left(\frac{1}{N}\right)\right) + \cdots + \frac{1}{N}f\left(1+\frac{i}{N}\right) + \cdots + \frac{1}{N}f\left(1+\frac{N}{N}\right)$$

$x_1 = 1 + \Delta x$

$x_2 = 1 + 2\Delta x$

$\vdots$

$x_i = 1 + i\Delta x$

(b) [6 points] Simplify the formula for R_N using the power sum formulas given on the last page, and your other knowledge of how to compute with summations.

$$R_N = \sum_{i=1}^{N} \frac{1}{N} f\left(1+\frac{i}{N}\right) = \sum_{i=1}^{N} \frac{1}{N}\left[2\left(1+\frac{i}{N}\right)-1\right] = \sum_{i=1}^{N} \frac{1}{N}\left[1+\frac{2i}{N}\right]$$

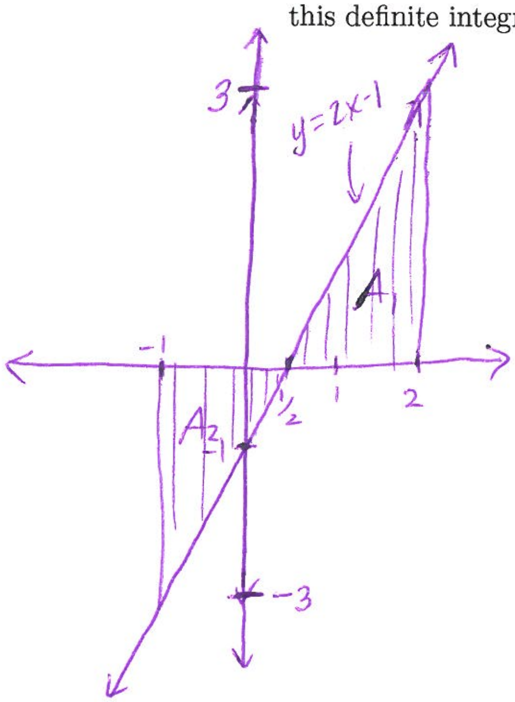
$$= \frac{1}{N}\sum_{i=1}^{N}(1) + \frac{1}{N}\sum_{i=1}^{N}\frac{2i}{N} = \frac{1}{N}(N) + \frac{2}{N^2}\sum_{i=1}^{N} i = 1 + \frac{2}{N^2}\left(\frac{N(N+1)}{2}\right)$$

(c) [8 points] Find the value of $\int_1^2 f(x)\,dx$ by evaluating $\lim_{N\to\infty} R_N$.

$$\int_1^2 f(x)\,dx = \lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(1 + \frac{\cancel{2}}{N^{\cancel{2}}}\left(\frac{\cancel{N}(N+1)}{\cancel{2}}\right)\right) = \lim_{N\to\infty}\left(1+\frac{N+1}{N}\right)$$

$$= \lim_{N\to\infty}\left(1+1+\frac{1}{N}\right) = \underline{2}.$$

(d) [6 points] Finally, below is a graph of f(x). Use geometry to compute $\int_{-1}^2 f(x)\,dx$. Show your work, or include an explanation for your result. (NOTICE the limits on this definite integral are different from the limits we had in the earlier parts!)



$$A_1 = \tfrac{1}{2}b\cdot h = \tfrac{1}{2}\cdot\left(1\tfrac{1}{2}\right)(3) = \tfrac{1}{2}\left(\tfrac{3}{2}\right)(3) = \tfrac{9}{4}$$

$$A_2 = \tfrac{1}{2}b\cdot h = \tfrac{1}{2}\left(1\tfrac{1}{2}\right)(3) = \tfrac{9}{4}$$

$$\int_{-1}^{2} f(x)\,dx = A_1 - A_2 = \frac{9}{4} - \frac{9}{4} = \underline{0}.$$

3. [10 points] Let $s(t)$ = position of an object at time t. If the velocity of the object is given by $v(t) = -t^2+4t+8$ meters per second, what is displacement of the object after 3 seconds, if the initial position of the object is $s(0) = 0$?

displacement after 3 secs $= s(3) - s(0)$

and: $s(3)-s(0) = \int_0^3 s'(t)\,dt = \int_0^3 v(t)\,dt = \int_0^3 -t^2+4t+8\,dt$

$$= -\frac{1}{3}t^3+2t^2+8t\Big|_0^3 = \underline{\underline{(-3^2+18+24) - 0}}\ \underline{m}$$

2. (a) [12 points] Complete the table of indefinite integrals.

$\int f(x)\,dx$	$F(x)+C$	$\int f(x)\,dx$	$F(x)+C$
$\int k\,dx$	$kx+C$	$\int x^n\,dx$	$\frac{x^{n+1}}{n+1}+C$
$\int \frac{1}{x}\,dx$	$\ln\lvert x\rvert+C$	$\int e^x\,dx$	e^x+C
$\int \sin(x)\,dx$	$-\cos(x)+C$	$\int \cos(x)\,dx$	$\sin(x)+C$
$\int \sec^2(x)\,dx$	$\tan(x)+C$	$\int \sec(x)\tan(x)\,dx$	$\sec(x)+C$
$\int b^x\,dx$	$(\ln(b))^{-1}b^x+C$	$\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx$	$\operatorname{arcsec}(x)+C$
$\int \frac{1}{\sqrt{1-x^2}}\,dx$	$\arcsin(x)+C$	$\int \frac{1}{x^2+1}\,dx$	$\arctan(x)+C$

(b) [8 points] Let $\int_1^x e^{\sin(t)}\,dt$. Find $A(1)$, $A'(x)$, and $A'(\pi)$.

given on board during exam $\rightarrow$ $\boxed{A(x)=\int_1^x e^{\sin(t)}\,dt}$ so $A(1)=\int_1^1 e^{\sin(t)}\,dt=0$

by FTC II, $A'(x)=e^{\sin(x)}$

so $A'(\pi)=e^{\sin(\pi)}$.

(c) [8 points] Compute $\frac{d}{dx}\left(\int_0^{\ln(x)} \frac{t}{t^3+2}\,dt\right)$.

Let $A(x)=\int_0^x \frac{t}{t^3+2}\,dt \Rightarrow A'(x)=\frac{x}{x^3+2}$ by FTC II

and $g(x)=\ln(x) \Rightarrow g'(x)=\frac{1}{x}$

so $\left(\int_0^{\ln(x)} \frac{t}{t^3+2}\,dt\right)' = \left(A(g(x))\right)' = A'(g(x))\cdot g'(x)$ by chain rule

$$= \underline{\left(\frac{\ln(x)}{(\ln(x))^3+2}\right)\cdot\frac{1}{x}}$$

4. Evaluate the following:

(a) [7 points]

$$\int \frac{2(x+3)^2}{x}\,dx = \int \frac{2(x^2+6x+9)}{x}\,dx$$

$$= 2\int x+6+\frac{9}{x}\,dx = 2\left(\frac{1}{2}x^2+6x+9\ln|x|\right)+C$$

(b) [7 points]

$$\int_1^2 t\cos(3t^2-4)\,dt$$

u-sub:

$$\begin{cases} u = 3t^2-4 \\ du = 6t\,dt \end{cases}$$

$t=1 \Rightarrow u = 3-4 = -1$

$t=2 \Rightarrow u = 12-4 = 8$

$$= \int_{-1}^{8} \frac{1}{6}\cos(u)\,du = \frac{1}{6}\sin(u)\Big|_{-1}^{8}$$

$$= \frac{1}{6}\left(\sin(8) - \sin(-1)\right).$$

(c) [8 points]

$$\int \frac{(\ln(x))^{3/2}}{x}\,dx$$

let $u = \ln(x)$

$\Rightarrow du = \frac{1}{x}\,dx$

$$= \int u^{3/2}\,du = \frac{2u^{5/2}}{5}+C = \frac{2}{5}(\ln(x))^{5/2}+C$$

(c) [8 points]

$$\int \frac{1}{4+25x^2}\,dx = \int \frac{1}{4+4u^2}\cdot\frac{2}{5}\,du$$

let $25x^2 = 4u^2$

$\Rightarrow 5x = 2u$

$\Rightarrow u = \frac{5}{2}x$

$\Rightarrow du = \frac{5}{2}dx$

$\Rightarrow \frac{2}{5}du = dx$

$$= \frac{2}{5}\cdot\frac{1}{4}\int\frac{1}{1+u^2}\,du$$

$$= \frac{2}{20}\arctan(u)+C$$

$$= \frac{1}{10}\arctan\left(\frac{5}{2}x\right)+C$$

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$$

$$\sum_{i=1}^{N} i^2 = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}$$

$$\sum_{i=1}^{N} i^3 = \frac{N^2(N+1)^2}{4}$$