Example: Arclength Approximation

Approximate the arclength of $f(x) = \cos(x)$ for x in $[0, \pi]$: (Use S_6)

$$s = \int_0^{\pi} \sqrt{1 + (-\sin(x))^2}\, dx = \int_0^{\pi} \sqrt{1 + \sin^2(x)}\, dx$$

$$\approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6) \right]$$

$$\underline{\Delta x} = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

$x_0 = 0,\ x_1 = \frac{\pi}{6},\ x_2 = \frac{\pi}{3},\ x_3 = \frac{\pi}{2},\ x_4 = \frac{2\pi}{3},\ x_5 = \frac{5\pi}{6},\ x_6 = \pi$

$$\approx \frac{\pi}{18}\left[f(0) + 4f\left(\tfrac{\pi}{6}\right) + 2f\left(\tfrac{\pi}{3}\right) + 4f\left(\tfrac{\pi}{2}\right) + 2f\left(\tfrac{2\pi}{3}\right) + 4f\left(\tfrac{5\pi}{6}\right) + f(\pi) \right]$$

$$\approx \frac{\pi}{18}\left[1 + \cancel{4}^{2}\sqrt{\frac{5}{\cancel{4}}} + 2\sqrt{\frac{7}{4}} + 4\sqrt{2} + 2\sqrt{\frac{7}{4}} + \cancel{4}^{2}\sqrt{\frac{5}{\cancel{4}}} + 1 \right]$$

$$\approx \frac{\pi}{18}\left[2 + 4\sqrt{5} + 2\sqrt{7} + 4\sqrt{2} \right] \approx \underline{3.820985}$$

approx arc length.

and

$$\text{error}(S_6) \le \frac{K_4(\pi - 0)^5}{180 \cdot 6^4} = \frac{K_4 \pi^5}{180 \cdot 6^4}$$

by <u>new</u> estimate for $K_5 = 7$

$$\Rightarrow \text{error}(S_6) \le \frac{7\pi^5}{180 \cdot 6^4} \approx 0.0092$$

So $\underline{\underline{\text{actual arclength } s}}$ is such that

$$3.820985 - 0.009215 \leq s \leq 3.820985 + 0.009215$$

$$\boxed{3.8118 \leq s \leq 3.8302}$$

finding K

$$f(x)=\sqrt{1+\sin^2(x)}$$

$$f'(x)=\frac{1}{2}\left(1+\sin^2(x)\right)^{-1/2}\left(\sin(2x)\right)$$

$$f''(x)=-\frac{1}{4}\left(1+\sin^2(x)\right)^{-3/2}\left(\sin(2x)\right)^2+\left(1+\sin^2(x)\right)^{-1/2}\left(\cos(2x)\right)$$

$$f'''(x)=\frac{3}{8}\left(1+\sin^2(x)\right)^{-5/2}\left(\sin(2x)\right)^3$$

$$-\left(1+\sin^2(x)\right)^{-3/2}\sin(2x)\cos(2x)-\frac{1}{2}\left(1+\sin^2(x)\right)^{-3/2}\sin(2x)\cos(2x$$

$$-2\left(1+\sin^2(x)\right)^{-1/2}\sin(2x)$$

$$=\frac{3}{8}\left(1+\sin^2(x)\right)^{-5/2}\sin^3(2x)$$

$$-\frac{3}{2}\left(1+\sin^2(x)\right)^{-3/2}\sin(2x)\cos(2x)-2\left(1+\sin^2(x)\right)^{-1/2}\sin(2x)$$

$$f^{(4)}(x)=\frac{-15}{16}\left(1+\sin^2(x)\right)^{-7/2}\sin^4(2x)+\frac{9}{4}\left(1+\sin^2(x)\right)^{-5/2}\sin^2(2x)\cos$$

$$+\frac{9}{4}\left(1+\sin^2(x)\right)^{-5/2}\sin^2(2x)\cos(2x)-3\left(1+\sin^2(x)\right)^{-3/2}\left[\cos^2(2x)-\sin^2(2x)\right]$$

$$+3\left(1+\sin^2(x)\right)^{-3/2}\sin^2(2x)-4\left(1+\sin^2(x)\right)^{-1/2}\cos(2x)$$

$$f^{(4)}(x) = -15\left(1+\sin^2(x)\right)^{-7/2}\sin^4(x)\cos^4(x)$$

$$+\ 18\left(1+\sin^2(x)\right)^{-5/2}\sin^2(x)\cos^2(x)\left(\cos^2(x)-\sin^2(x)\right)$$

$$-\ 3\left(1+\sin^2(x)\right)^{-3/2}\left[\cos^4(x) - \cancel{6\cos^2(x)\sin^2(x)} + \sin^4(x)\right]$$

$$+\ 4\left(1+\sin^2(x)\right)^{-3/2}\sin^2(x)\cos^2(x) - 4\left(1+\sin^2(x)\right)^{-1/2}\left(\cos^2(x) - \sin^2(x)\right)$$

$$= \left(1+\sin^2(x)\right)^{-7/2}\Big[-15\cancel{\sin^4(x)\cos^4(x)} + 18\left(\cancel{\sin^2\cos^4} - \cancel{\sin^4\cos^2}\right)$$

$-18+8-12$

$$+\ 18\left(\cancel{\sin^4\cos^4} - \cancel{\sin^6\cos^2}\right)$$

$$-\ 3\left(\cancel{\cos^4} + \cancel{\sin^4} + 2\cancel{\sin^2\cos^4} + 2\cancel{\sin^6} + \cancel{\cos^4\sin^4} + \cancel{\sin^8}\right)$$

72

$$+\ 4\left(\cancel{\sin^2\cos^2} + 2\cancel{\sin^4\cos^2} + \cancel{\sin^6\cos^2}\right)$$

$132-90=42$

$$-\ 4\left(\cancel{1+3\sin^2+3\sin^4+\sin^6}\right)\left(\cancel{\cos^2-\sin^2}\right)\Big]$$

$$\left(\cos^2-\sin^2+\cancel{3\sin^2\cos^2}-\cancel{3\sin^4}+3\cancel{\sin^4\cos^2} - \cancel{3\sin^6}+\cancel{\sin^6\cos^2}-\cancel{\sin^8}\right)$$

$66-12$

$$= \left(1+\sin^2(x)\right)^{-9/2}\Big[12\sin^2\cos^4 - 22\sin^4\cos^2 - 18\sin^6\cos^2$$

$$-3\cos^4 + 9\sin^4 + 6\sin^6 + 5\sin^8 - 8\cos^2\sin^2$$

$$-4\cos^2 + 4\sin^2.$$

$$= (1+\sin^2(x))^{-7/2}\Big[2\sin^2\cos^2\big[\overset{-11+17\cos^2}{\cancel{6\cos^2-11\sin^2}}\big]$$
$$6\sin^6\big[\underset{-2+3\sin^2}{\cancel{-3\cos^2+1}}\big]$$
$$+{-4}+8\sin^2-8\cos^2\sin^2$$
$$-3+6\sin^2+6\sin^4$$
$$+\sin^8\Big]$$

$$= (1+\sin^2(x))^{-7/2}\Big[-22\sin^2\overset{1-\sin^2}{\cos^2} + 34\sin^2\overset{(1-2\sin^2+\sin^4)}{\cos^4}$$
$$-12\sin^6+19\sin^8$$
$$-4+14\sin^4$$
$$-3+6\sin^2\Big]$$

$\sin^6(-12+19\sin^2)$
$17-19\cos^2$

-28

$$= (1+\sin^2(x))^{-7/2}\Big[-7-32\sin^4(x)+18\sin^2(x) + 22\sin^6(x)+19\sin^8(x)\Big]$$

most negative? for $\sin x=0 \Rightarrow$ denom is as small as possible.

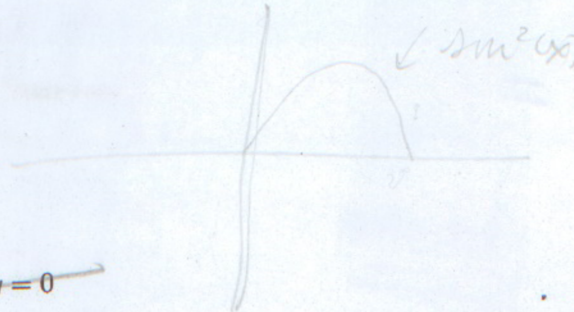
and num $= -7$.

for $\sin x = \pm 1$, won't be neg! and divide by number larger than 11. $(2^{7/2})$ so will never be > 7 in abs value.

for intermediate values w/ $0 < |\sin(x)| < 1$,

extrema at $\approx 21.26°, 90°, 158.74° \Rightarrow x = 0.6759, \frac{\pi}{2}, \pi - 0.6759$

$f^{(4)}(\frac{\pi}{2}) = \frac{20}{2^{7/2}} \Rightarrow f^{(4)}(0.7422) \approx 2$ so $\underline{K = 7}$.



4. (a) [8 points] Solve $t^2y'' + ty' - 9y = 0$.

(b) [9 points] Use the solution from part (a) and variation of paramenters to solve $t^2y'' + ty' - 9y = t$. (Note: if you could not solve part (a), come see me. I will give you a similar problem to solve).

$$-7 - 32\sin^4(x) + 18\sin^2(x)$$
$$+22\sin^6(x) + 19\sin^8(x)$$

finding extrema of $f^{(4)}(x)$.

$$g(x) = -32\sin^2(x) + 18 + 22\sin^4(x) + 19\sin^6(x)$$

$$g'(x) = -64\sin(x)\cos(x) + 88\sin^3(x)\cos(x) + 114\sin^5(x)\cos(x) = 0$$

$$g'(x) = \underbrace{2\sin(x)\cos(x)}_{x = 0, \pi, \pi/2}\left[\underbrace{-32 + 44\sin^2(x) + 57\sin^4(x)}\right] = 0$$

$$(1 + 7\sin^2(x))^2 = 1 + 4\sin^2 + 49\sin^4(x)$$

need
$$\left(\sin^2 + \frac{22}{57}\right) > .8428$$
for increasing.

$$57\left[\left(\sin^4(x) + \frac{44}{57}\sin^2 + \left(\frac{22}{57}\right)^2\right) - \left(\frac{32}{57} + \frac{22^2}{57^2}\right)\right]$$

$$\left(\sin^2 + \frac{22}{57}\right)^2 - \left(\frac{32}{57} + \frac{22^2}{57^2}\right) \quad .7104$$

$\sin^2 \cancel{+\frac{22}{57}}$ 0.4568

inc when $\sin^2 > \pm\sqrt{\frac{32}{57} + \frac{22^2}{57^2}} - \frac{22}{57}$

$\sin > 0.6759 \Rightarrow 21.26°$ & $\sin < 180 - 21.26°$