Solutions

Math 142
The University of Tennessee

Spring Semester 2016
H. Finotti

Practice Problems 1

1. This problem will walk you step by step through computing the area under a curve via the limit of an approximation by rectangles. For this exercise, we want first the approximate area, then the exact area under the function

$$f(x) = x^3 - x$$

over the interval $[1, 2]$.

(a) First, we need to subdivide the interval $[1, 2]$ into N subintervals of equal length. Find the length of each subinterval and the coordinates of the endpoints of each subinterval.

$\Delta x = \frac{2-1}{N} = \frac{1}{N}$

$x_0 = 1$ $\quad x_2 = 1 + \frac{2}{N}$

$x_1 = 1 + \frac{1}{N}$ $\quad x_3 = 1 + \frac{3}{N}$ $\quad \cdots \quad \underline{x_i} = 1 + \frac{i}{N}$ = right endpoint of ith subinterval

(b) Second, choose between finding R_N, L_N, and M_N to find the approximate area under the curve via areas of rectangles over the subintervals. Once you make your choice, fill in the results below:

The area of the first subrectangle is: $\Delta x f(x_1) = \left(\frac{1}{N}\right)\left(\left(1+\frac{1}{N}\right)^3 - \left(1+\frac{1}{N}\right)\right)$

The area of the second subrectangle is: $\Delta x f(x_2) = \left(\frac{1}{N}\right)\left(\left(1+\frac{2}{N}\right)^3 - \left(1+\frac{2}{N}\right)\right)$

The area of the third subrectangle is: $\Delta x f(x_3) = \left(\frac{1}{N}\right)\left(\left(1+\frac{3}{N}\right)^3 - \left(1+\frac{3}{N}\right)\right)$

The area of a general i^{th} subrectangle is: $\Delta x f(x_i) = \left(\frac{1}{N}\right)\left(\left(1+\frac{i}{N}\right)^3 - \left(1+\frac{i}{N}\right)\right)$

The area of the *last* subrectangle is: $\Delta x f(x_N) = \left(\frac{1}{N}\right)\left(2^3 - 2\right)$

(c) Third, using what you found above write out the summation that will give us the approximate area under the curve. First write it out term by term, and then use summation notation to write it in a more compact form.

$$R_N \approx \text{area} \approx \left(\frac{1}{N}\right)\left(\left(1+\frac{1}{N}\right)^3 - \left(1+\frac{1}{N}\right)\right) + \left(\frac{1}{N}\right)\left(\left(1+\frac{2}{N}\right)^3 - \left(1+\frac{2}{N}\right)\right) + \cdots + \left(\frac{1}{N}\right)\left(2^3 - 2\right)$$

$$\approx \sum_{i=1}^{N} \left(\frac{1}{N}\right)\left(\left(1+\frac{i}{N}\right)^3 - \left(1+\frac{i}{N}\right)\right)$$

(d) Fourth, using the power sum equalities in the text, rewrite R_N, L_N, or M_N (depending on which you did above) in terms of N only. At this point we want to eliminate the summation notation entirely and end up with a simpler expressions that depends only on N.

$$R_N = \sum_{i=1}^{N} \left(\frac{1}{N}\right)\left(\left(1+\frac{i}{N}\right)^3 - \left(1+\frac{i}{N}\right)\right) = \sum_{i=1}^{N} \frac{1}{N}\left[\left(1+\frac{i}{N}\right)\left(\left(1+\frac{i}{N}\right)^2 - 1\right)\right]$$

$$= \sum_{i=1}^{N} \frac{1}{N}\left(1+\frac{i}{N}\right)\left[\cancel{1} + \frac{2i}{N} + \frac{i^2}{N^2} \cancel{-1}\right] = \sum_{i=1}^{N} \frac{1}{N}\left[\frac{2i}{N} + \frac{i^2}{N^2} + \frac{2i^2}{N^2} + \frac{i^3}{N^3}\right]$$

$$= \frac{2}{N^2}\sum_{i=1}^{N} i + \frac{3}{N^3}\sum_{i=1}^{N} i^2 + \frac{1}{N^4}\sum_{i=1}^{N} i^3 = \frac{2}{N^2}\left(\frac{N(N+1)}{2}\right) + \frac{3}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) + \frac{1}{N^4}\left(\frac{N^2(N+1)^2}{4}\right)$$

(e) Use your result in part (d) to find the approximate area under the curve using $N = 10$ rectangles.

Since simplifying above gives

$$\boxed{R_N = 1 + \frac{1}{N} + 1 + \frac{3}{2N} + \frac{3}{6N^2} + \frac{1}{4} + \frac{1}{2N} + \frac{1}{4N^2}}$$

$$\Rightarrow R_{10} = 2\tfrac{1}{4} + \frac{1}{10} + \frac{3}{20} + \frac{3}{600} + \frac{1}{20} + \frac{1}{400}$$ ← approx area via 10 rectangles.

$$= 2\tfrac{1}{4} + .3075 = 2.5575$$

(f) Use your result in part (d) to find the *exact* area under the curve, by taking the limit of the expression as $N \to \infty$. Make sure you show your work for computing the limit.

$$\text{exact area} = \lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(2\tfrac{1}{4} + \frac{1}{N} + \frac{3}{2N} + \frac{1}{2N} + \frac{1}{4N^2} + \frac{3}{6N^2}\right)$$

$$= 2\tfrac{1}{4} = \underline{2.25}.$$