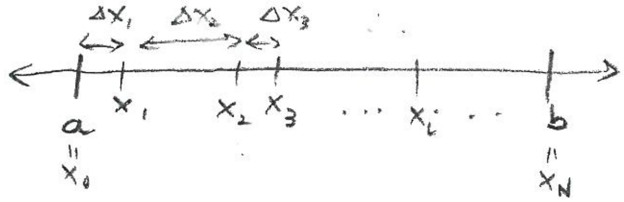
* Why does the FTC I work?? *

① If f is continuous on $[a,b]$, then f is integrable on $[a,b]$

meaning:

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i$$

exists and yields the same value regardless of the choice of partition P or sample points c_i.

② 

← take any partition of $[a,b]$ with N subintervals.

$$\Rightarrow F(b)-F(a) = \underbrace{F(b)-F(x_{N-1})}+\underbrace{F(x_{N-1})-F(x_{N-2})}+F(x_{N-2})+\cdots$$

$$\cdots+\underbrace{F(x_2)-F(x_1)}+\underbrace{F(x_1)-F(a)}$$

$$=\left(\frac{F(b)-F(x_{N-1})}{\Delta x_N}\right)\Delta x_N+\left(\frac{F(x_{N-1})-F(x_{N-2})}{\Delta x_{N-1}}\right)\Delta x_{N-1}+\cdots+\left(\frac{F(x_2)-F(x_1)}{\Delta x_2}\right)\Delta x_2$$

$$+\left(\frac{F(x_1)-F(a)}{\Delta x_1}\right)\Delta x_1$$

by the mean value theorem, we know on each subinterval $[x_i, x_{i+1}]$ there is some point c_i such that $\dfrac{F(x_{i+1})-F(x_i)}{\Delta x_i} = F'(c_i)$ (i.e. slope of secant thru endpts = slope of tangent at c_i)

so our sum is the same as

$$= F'(c_N)\Delta x_N + F'(c_{N-1})\Delta x_{N-1}+\cdots+F'(c_2)\Delta x_2+F'(c_1)\Delta x_1$$

since $F'=f$
$$= f(c_N)\Delta x_N + f(c_{N-1})\Delta x_{N-1}+\cdots+f(c_2)\Delta x_2+f(c_1)\Delta x_1$$

$$=\sum_{i=1}^{N} f(c_i)\Delta x_i$$ and we've shown that

for *this* *special* choice of sample points c_i,
we actually get

$$F(b)-F(a) = \sum_{i=1}^{N} f(c_i)\Delta x_i .$$

If we take the limit of both sides as $N \to \infty$,
we have

$$\boxed{\underbrace{F(b)-F(a)}_{\substack{\text{b/c it is constant} \\ \text{with respect to } N}} = \lim_{N\to\infty} \sum_{i=1}^{N} f(c_i)\Delta x_i = \int_a^b f(x)\,dx}$$

because f is *integrable*, so the choice of partition + c_i's don't matter!

which is exactly the conclusion of the FTC I.!