Leslie Matrix Models

November 3, 2013

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Long Term Growth Rates (Eigenvalues!)

- We have already seen that we cannot necessarily expect our population to stabilize long terms in terms of it's size.
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- Leslie matrices ALWAYS have exactly one positive dominant eigenvalue.

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- This is exactly what we saw in the examples we did.

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- But FIRST, we need to learn how to find a determinant...

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Recall that the simple wetland model was:

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Another Example - Eigenvalues of a Leslie Matrix

The long term growth rate of the Leslie matrix

$$A = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}$$

is given by it's dominant eigenvalue:

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