A PROBLEM OF ZAGIER ON QUADRATIC POLYNOMIALS AND CONTINUED FRACTIONS

MARIE JAMESON

ABSTRACT. For non-square $1 < D \equiv 0, 1 \pmod 4$, Zagier [9] defined the following summatory function using integral quadratic polynomials:

$$A_D(x) := \sum_{\substack{\mathrm{disc}(Q)=D \\ Q(\infty)<0<Q(x)}} Q(x).$$

He proved that $A_D(x)$ is a constant function depending on D. For rational x, it turns out that this sum is finite. Here we address the infinitude of the number of quadratic polynomials for nonrational x, and more importantly address some problems posed by Zagier related to characterizing the polynomials which arise in terms of the continued fraction expansion of x. In addition, we study the indivisibility of the constant functions $A_D(x)$ as D varies.

1. Introduction and Statement of Results

Following Zagier [9], we consider the function $A_D(x)$ defined as follows: for any real number x and any positive non-square integer D which is congruent to 0 or 1 modulo 4, consider all quadratic polynomials with integer coefficients and discriminant D which are negative at infinity and positive at x. For any such quadratic function Q, we have that $Q(x)$ is positive and wish to find the sum of these values. That is, we consider the function

$$A_D(x) := \sum_{\substack{\mathrm{disc}(Q)=D \\ Q(\infty)<0<Q(x)}} Q(x). \tag{1.1}$$

It is known that the function $A_D(x)$ is determined by its behavior for $x \in [0, 1)$ (see Lemma 2.1), so we shall always assume that $0 \leq x < 1$. For example, when $x = 0$ and $D = 5$, there are only two quadratic polynomials with the desired properties: $Q(X) = -X^2 + X + 1$ and $Q(X) = -X^2 - X + 1$, giving $A_5(0) = 1 + 1 = 2$. It turns out that much more is true about these functions. Zagier [9] proved that each function $A_D(x)$ is constant (although the polynomials which arise in the sum vary with x). In particular, we have the strange fact that $A_5(1/\pi) = A_5(0) = 2$. Notice then that for $x = 1/\pi$, there must be infinitely many quadratic polynomials in the sum, since $1/\pi$ is irrational and does not have degree 2 over $\mathbb{Q}$.

In this paper, we address the following natural question regarding the function $A_D(x)$: given a value of x, how can we characterize the quadratic polynomials with the desired properties? In [9], Zagier investigated this question, and he made a speculation which involves quantities which arise from the continued fraction expansion of x.

To make this precise, we must first fix some notation. For x a real number with $0 < x < 1$, we may write x as a continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [0; a_1, a_2, \ldots]$$

using integers $a_1, a_2, \ldots \geq 1$. Note that this continued fraction terminates if and only if $x \in \mathbb{Q}$. As in [9], we now define a useful sequence of real numbers $\delta_0, \delta_1, \ldots$ by

$$\delta_0 = 1, \qquad \delta_1 = x, \qquad \delta_{n+1} = \delta_{n-1} - a_n \delta_n \qquad (n \geq 1).$$

Zagier made the following speculation based on numerical evidence for $D = 5$ and $x = \frac{1}{\pi}$:

Speculation. *Suppose that $D = 5$ and $0 < x < 1$. Then the summands which appear on the right side of (1.1) are: all of the expressions*

$$-\delta_{n+1}^2 + \delta_n \delta_{n+1} + \delta_n^2$$

together with some of the expressions

$$-\delta_{n+1}^2 - \delta_n \delta_{n+1} + \delta_n^2.$$

Of course, answering this question amounts to characterizing the set of polynomials

$$\Omega_D(x) := \{aX^2 + bX + c : D = b^2 - 4ac, a < 0 < ax^2 + bx + c\}. \tag{1.2}$$

Here we offer a theorem which characterizes $\Omega_D(x)$. In Section 2.3, we define sets of 4-tuples $\Omega_D^0(x)$ and corresponding quadratic polynomials $\psi(a, b, c, n; X)$, and we prove the following theorem.

Theorem 1.1. *Fix a real number x with $0 < x < 1$, and a positive integer $D \equiv 0, 1 \pmod 4$. If D is not a square, we have*

$$\Omega_D(x) = \{\psi(a, b, c, n; X) : (a, b, c, n) \in \Omega_D^0(x)\}.$$

If $D = m^2$ for some positive integer m, we have

$$\begin{aligned}\Omega_D(x) = &\{\psi(a, b, c, n; X) : (a, b, c, n) \in \Omega_D^0(x)\}\\ &\cup \{\psi(-a, m, 0, n; X) : n \geq 0 \text{ and } 1 \leq a \leq a_{n+1} m\}.\end{aligned}$$

Remark. As discussed in [9], $A_D(x)$ can also be defined when for square $D = m^2$. In that case, we define $A_{m^2}^*(x)$ to be the sum in (1.1), and set

$$A_{m^2}(x) := A_{m^2}^*(x) - \frac{1}{2}\overline{\mathbb{B}}_2(mx) + \frac{1}{2}m^2 \kappa(x),$$

where $\mathbb{B}_2(x) := x^2 - x + \frac{1}{6}$ is the second Bernoulli polynomial, $\overline{\mathbb{B}}_2(x) := \mathbb{B}_2(x - \lfloor x \rfloor)$, and

$$\kappa(x) := \begin{cases} 1/q^2 & x = p/q \text{ with } (p, q) = 1 \\ 0 & x \text{ is irrational} \end{cases}.$$

This theorem provides the following corollary.

Corollary 1.2. *For $x \in \mathbb{R}$ and D as above, we have that*

$$\#\Omega_D(x) < +\infty \iff x \in \mathbb{Q}.$$

Remark. By a result of Zagier (Theorem 1 of [9]) which states that $A_D(x)$ is a rational constant, this is trivial except for x such that $[\mathbb{Q}(x):\mathbb{Q}]=2$ (see Lemma 2.3).

The description of $\Omega_D^0(x)$ given in Section 2.3 when $D=5$ will show that we have indeed established Zagier's speculation. Namely, we have the following corollary.

Corollary 1.3. *Suppose that $D=5$ and $0<x<1$. Then the summands which appear on the right side of (1.1) are all of the expressions*

$$-\delta_{n+1}^2+\delta_n\delta_{n+1}+\delta_n^2$$

together with some of the expressions

$$-\delta_{n+1}^2-\delta_n\delta_{n+1}+\delta_n^2.$$

Furthermore, if $a_n\neq 1$ and $a_{n+1}\neq 1$ for a value of n, then the expression $-\delta_{n+1}^2-\delta_n\delta_{n+1}+\delta_n^2$ does appear as a summand.

Remark. It is natural to wonder what the generalization of Zagier's speculation should be for other D. We will show that for non-square D, the summands which appear are of the form $a\delta_{n+1}^2+b\delta_n\delta_{n+1}+c\delta_n^2$, where $aX^2+bX+c\in\Omega_D(0)$. Furthermore, if $aX^2+bX+c\in\Omega_D(0)$ comes from a *reduced* binary quadratic form, then *all* of the terms $a\delta_{n+1}^2+b\delta_n\delta_{n+1}+c\delta_n^2$ appear in the sum.

By Zagier's theorem, we know that each function $A_D(x)$ is a rational constant which depends only on D. Here are the first few constant functions $A_D(x)$ for non-square D.

D	5	8	12	13	17	20	21
$A_D(x)$	2	5	10	10	20	22	20

It is natural to wonder about their properties as D varies. Here we study the distribution of these numbers modulo primes ℓ, and we prove the following theorem using the theory of Cohen-Eisenstein series.

Theorem 1.4. *Suppose that $\ell>5$ is prime, and let p be any prime for which $p\equiv -1 \pmod{\ell}$ and $p\equiv 2,3 \pmod 5$. Then there exists an integer $1\le n_p\le \frac{5}{4}(p+1)$ for which $A_{pn_p}(x)\not\equiv 0 \pmod{\ell}$.*

As a corollary, we obtain the following.

Corollary 1.5. *If $\ell>5$ is prime and $\epsilon>0$, then for all sufficiently large X we have that*

$$\#\{0<D\equiv 0,1 \pmod 4 \le X \ :\ \ell\nmid A_D(x)\}\ge\left(\frac{1}{\sqrt{5}(\ell-1)}-\epsilon\right)\frac{\sqrt{X}}{\log X}.$$

2. Nuts and Bolts

Before we prove Theorem 1.1 and its corollaries, we must first recall some basic facts and definitions regarding $A_D(x),\Omega_D(x)$, and continued fractions. We will then use Zagier's speculation as a model to define a helpful function $\psi(a,b,c,n;X)$ and various sets $\Omega_D^i(x)$.

2.1. **Background on Continued Fractions.** First we recall some classical facts regarding continued fractions. The following facts can be found in [4] or Section 10 of [9]. Recall that for any real number x with $0 < x < 1$, we may write x as a continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [0; a_1, a_2, \ldots]$$

using integers $a_1, a_2, \ldots \geq 1$, and that this continued fraction terminates if and only if $x \in \mathbb{Q}$. The convergents

$$\frac{p_n}{q_n} = [0; a_1, \ldots, a_n]$$

of the continued fraction are given by: $p_{-1} = 1, q_{-1} = 0, p_0 = 0, q_0 = 1$, and

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \\ q_n &= a_n q_{n-1} + q_{n-2}. \end{aligned}$$

It is known that the value of x is greater than that of any even order convergent p_n/q_n, and less than that of any odd order convergent, and for all $n \geq 0$, we have

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n \tag{2.1}$$

(see Theorems 8 and 2 of [4]).

We have defined $\delta_0, \delta_1, \ldots$ by

$$\delta_0 = 1, \qquad \delta_1 = x, \qquad \delta_{n+1} = \delta_{n-1} - a_n \delta_n \qquad (n \geq 1).$$

One can check that

$$\delta_{n+1} = |p_n - q_n x|,$$

that $1 = \delta_0 > \delta_1 > \delta_2 > \cdots > 0$, and that

$$\frac{\delta_n}{\delta_{n-1}} = [0; a_n, a_{n+1}, \ldots].$$

2.2. **Elementary Facts about $A_D(x)$ and $\Omega_D(x)$.** Here we state some important properties of $A_D(x)$ and $\Omega_D(x)$. All of the results in this section are contained in [9]. First we have the following elementary observation.

Lemma 2.1. *For any real number x and any positive integer D which is congruent to 0 or 1 modulo 4, we have that*

$$A_D(x) = A_D(x+1).$$

Proof. First suppose that D is not a square. We have that

$$\begin{aligned}
A_D(x) &= \sum_{\substack{a,b,c\in\mathbb{Z}\\ b^2-4ac=D\\ a<0<ax^2+bx+c}} \big(ax^2+bx+c\big)\\
&= \sum_{\substack{a,b,c\in\mathbb{Z}\\ (b-2a)^2-4a(a-b+c)=D\\ a<0<a(x+1)^2+(b-2a)(x+1)+(a-b+c)}} \big(a(x+1)^2+(b-2a)(x+1)+(a-b+c)\big)\\
&= \sum_{\substack{\alpha,\beta,\gamma\in\mathbb{Z}\\ \beta^2-4\alpha\gamma=D\\ \alpha<0<\alpha(x+1)^2+\beta(x+1)+\gamma}} \big(\alpha(x+1)^2+\beta(x+1)+\gamma\big)\\
&= A_D(x+1),
\end{aligned}$$

as desired. If D is a square, the proof follows similarly. □

Next, we recall a deeper theorem of Zagier:

Lemma 2.2 (Theorem 1 and Supplement to Theorem 1 of [9]). *For D as described above, the function $A_D(x)$ has a constant rational value which we denote α_D. If D is the discriminant of a real quadratic field, we have that*

$$A_D(x) = \alpha_D = -5L(-1,\chi_D).$$

Remark. In fact, Zagier [9] described α_D in terms of the coefficients of the weight $\frac{5}{2}$ Cohen-Eisenstein series $H_2(z)$ discussed in Section 5.

Finally, we summarize previous results regarding $\#\Omega_D(x)$.

Lemma 2.3. *For x and D as described above, the following are true:*
(a) If $x \in \mathbb{Q}$, then $\#\Omega_D(x) < +\infty$.
(b) If $x \in \mathbb{R}\setminus\mathbb{Q}$ and x is not algebraic of degree 2 over $\mathbb{Q}$, then $\#\Omega_D(x) = +\infty$.

Proof. First we prove (a). If $x\in\mathbb{Q}$, then we may write $x = p/q$ and note that if $aX^2+bX+c \in \Omega_D(x)$, then we have

$$Dq^2 = |bq+2ap|^2 + 4|a||ap^2+bpq+cq^2|.$$

This bounds each of a, b, and c, so $\#\Omega_D(x) < +\infty$ (note: this corrects a typo in [9]).

To prove (b), let $x \in \mathbb{R}\setminus\mathbb{Q}$ and suppose that x is not algebraic of degree 2 over $\mathbb{Q}$, and let D as above be non-square (if D is a square, then the proof follows similarly). Suppose for contradiction that $\#\Omega_D(x) < +\infty$. Then since $A_D(x)$ has a constant integral value, one can solve

$$\sum_{\substack{a,b,c\in\mathbb{Z}\\ b^2-4ac=D\\ a<0<ax^2+bx+c}} \big(ax^2+bx+c\big) = A_D(x)$$

to find that x is the root of a quadratic polynomial (this equation cannot be trivial since each polynomial has negative leading coefficient). This contradicts our choice of x. □

2.3. **Defining** $\psi(a,b,c,n;X)$ **and** $\Omega^0_D(x)$**.** Let us explicitly write down the polynomials which Zagier has mentioned: since

$$\delta_n = |p_{n-1} - q_{n-1}x| = (-1)^n(p_{n-1} - q_{n-1}x)$$

by Theorem 8 of [4], we may substitute to find that these expressions from Zagier's speculation (and Corollary 1.3) can be written as the values of the polynomials

$$-(p_n - q_nX)^2 \mp (p_n - q_nX)(p_{n-1} - q_{n-1}X) + (p_{n-1} - q_{n-1}X)^2$$

when we plug in the value x for the variable X.

Now we extend this speculation as follows: for $0 < x < 1$ and $D \equiv 0, 1 \pmod 4$, we consider polynomials $aX^2 + bX + c \in \Omega_D(0)$ and nonnegative integers $n \geq 0$ and use them to build polynomials of the form

$$a(p_n - q_nX)^2 - b(p_n - q_nX)(p_{n-1} - q_{n-1}X) + c(p_{n-1} - q_{n-1}X)^2.$$

For ease of notation, we write

$$\Omega'_D := \{(a,b,c,n) \in \mathbb{Z}^4 : b^2 - 4ac = D, a < 0 < c, n \geq 0\}. \tag{2.2}$$

For $(a,b,c,n) \in \mathbb{Z}^3 \times \mathbb{Z}_{\geq 0}$, we build the polynomial

$$\begin{aligned}\psi(a,b,c,n;X) :=& a(p_n - q_nX)^2 - b(p_n - q_nX)(p_{n-1} - q_{n-1}X) + c(p_{n-1} - q_{n-1}X)^2 \\ =& \left(aX^2 + bX + c\right) | \begin{pmatrix} -q_n & p_n \\ q_{n-1} & -p_{n-1} \end{pmatrix},\end{aligned}$$

noting that $\psi(a,b,c,n;x) = a\delta^2_{n+1} + b\delta_{n+1}\delta_n + c\delta^2_n$. Here, the slash operator is defined by $f(X)| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (cX+d)^2 f\left(\frac{aX+b}{cX+d}\right)$ for quadratic polynomials f and 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Here, we must make the following remark regarding the case where $x \in \mathbb{Q}$. Since the number of polynomials in $\Omega_D(x)$ is finite if $x \in \mathbb{Q}$, rational values of x are less interesting than irrational values. However, the arguments in this paper hold for $x \in \mathbb{Q}$ as well as for $x \notin \mathbb{Q}$ (unless otherwise noted). One must be careful in only one regard: if $x = p/q$ is a rational number between 0 and 1, then its continued fraction expansion terminates, so $x = [0; a_1, a_2, \ldots, a_N]$, for positive integers N and $a_1, \ldots, a_N$. Thus we can only define finitely many convergents

$$\frac{p_{-1}}{q_{-1}}, \frac{p_0}{q_0}, \ldots, \frac{p_N}{q_N},$$

noting that $p_N/q_N = x$. We also have finitely many $\delta_0, \delta_1, \ldots, \delta_N, \delta_{N+1}$, noting that $\delta_N = 1/q^2$ and $\delta_{N+1} = 0$. Thus, when considering the case where $x \in \mathbb{Q}$, one must amend the arguments which follow by restricting his attention only to values which "make sense" (for example, only consider $\psi(a,b,c,n;X)$ for $n \leq m$). Thus, for simplicity of exposition, we will henceforth only describe the case where $x \notin \mathbb{Q}$, and leave rational values of x to the reader.

At first glance, it seems correct to consider the polynomials $\psi(a,b,c,n;X)$ for $(a,b,c,n) \in \Omega'_D$ since adding up the resulting values gives $A_D(x)$ (for non-square D) as desired:

$$\sum_{(a,b,c,n)\in\Omega'_D} \psi(a,b,c,n;x) = \sum_{\substack{(a,b,c)\in\mathbb{Z}^3\\ a<0<c\\ b^2-4ac=D}} \sum_{n\geq 0} \left(a\delta_{n+1}^2 + b\delta_{n+1}\delta_n + c\delta_n^2\right) = \sum_{\substack{(a,b,c)\in\mathbb{Z}^3\\ a<0<c\\ b^2-4ac=D}} \sum_{n\geq 0} \left(a\delta_{n+1}^2 + c\delta_n^2\right)$$

$$= \sum_{\substack{(a,b,c)\in\mathbb{Z}^3\\ a<0<c\\ b^2-4ac=D\\ -a<\sqrt{\frac{D-b^2}{4}}<c}} \sum_{n\geq 0} \left((a\delta_{n+1}^2 + c\delta_n^2) + (-c\delta_{n+1}^2 - a\delta_n^2)\right) + \sum_{\substack{(a,b,c)\in\mathbb{Z}^3\\ a<0<c\\ b^2-4ac=D\\ -a=c=\sqrt{\frac{D-b^2}{4}}}} \sum_{n\geq 0} \left(a\delta_{n+1}^2 + c\delta_n^2\right)$$

$$= \sum_{\substack{(a,b,c)\in\mathbb{Z}^3\\ a<0<c\\ b^2-4ac=D\\ -a<\sqrt{\frac{D-b^2}{4}}<c}} \sum_{n\geq 0} (c-a)(\delta_n^2 - \delta_{n+1}^2) + \sum_{\substack{(a,b,c)\in\mathbb{Z}^3\\ a<0<c\\ b^2-4ac=D\\ -a=c=\sqrt{\frac{D-b^2}{4}}}} \sum_{n\geq 0} c(\delta_n^2 - \delta_{n+1}^2)$$

$$= \sum_{\substack{(a,b,c)\in\mathbb{Z}^3\\ a<0<c\\ b^2-4ac=D\\ -a<\sqrt{\frac{D-b^2}{4}}<c}} (c-a) + \sum_{\substack{(a,b,c)\in\mathbb{Z}^3\\ a<0<c\\ b^2-4ac=D\\ -a=c=\sqrt{\frac{D-b^2}{4}}}} c = \sum_{\substack{(a,b,c)\in\mathbb{Z}^3\\ a<0<c\\ b^2-4ac=D}} c = A_D(0) = A_D(x).$$

However, the story is not so simple; as Zagier notes in [9], only *some* of the $\psi(a,b,c,n;x)$ actually appear as summands on the right hand side of (1.1). In fact, if $(a,b,c,n) \in \Omega'_D$, then one can easily use (2.1) to check that $\psi(a,b,c,n;X)$ has discriminant D, but it is not necessarily true that $\psi(a,b,c,n;X)_2 < 0$ or that $\psi(a,b,c,n;x) > 0$ as one would require, or that $\psi(a,b,c,n;X)$ is distinct from other polynomials of the same form. Here, $\psi(a,b,c,n;X)_2$ denotes the coefficient of X^2 in the polynomial $\psi(a,b,c,n;X)$.

Thus we define $\Omega_D^0(x) \subset \Omega'_D$ by

$$\Omega_D^0(x) := \left\{ (a,b,c,n) \in \Omega'_D : \begin{array}{c} \psi(a,b,c,n;X)_2 < 0 < \psi(a,b,c,n;x), \text{ and} \\ \psi(a,b,c,n;X) \neq \psi(\alpha,\beta,\gamma,m;X) \\ \text{for all } (\alpha,\beta,\gamma,m) \in \Omega'_D \text{ with } m > n \end{array} \right\}.$$

First note that for fixed n, all of the polynomials of the form $\psi(a,b,c,n;X)$ are distinct since we have the following:

Lemma 2.4. *If* $(a,b,c,n), (\alpha,\beta,\gamma,n) \in \mathbb{Z}^3 \times \mathbb{Z}_{\geq 0}$ *satisfy*

$$\psi(a,b,c,n;X) = \psi(\alpha,\beta,\gamma,n;X),$$

then $(a,b,c,n) = (\alpha,\beta,\gamma,n)$.

Proof. Suppose that $\psi(a,b,c,n;X) = \psi(\alpha,\beta,\gamma,n;X)$. Then substituting p_n/q_n for X gives $c = \gamma$ (since $p_{n-1} - q_{n-1}p_n/q_n \neq 0$ by (2.1)). Similarly, $a = \alpha$, so it follows that $b = \beta$ as desired. ∎

Also note that we have

$$\{\psi(a,b,c,n;X) : (a,b,c,n) \in \Omega_D^0(x)\} \subseteq \Omega_D(x)$$

by construction, and Theorem 1.1 asserts that this is an equality for non-square D.

2.4. **A Useful Partition of $\Omega'_D \setminus \Omega^0_D(x)$.** In order to prove Theorem 1.1, we will develop a better understanding of the behavior of $\psi(a,b,c,n;X)$ for $(a,b,c,n) \in \Omega'_D \setminus \Omega^0_D(x)$. Thus we will study the sets

$$\Omega^1_D(x) := \left\{ (a,b,c,n) \in \Omega'_D : \begin{array}{c} \psi(a,b,c,n;X)_2 < 0 < \psi(a,b,c,n;x), \text{ and} \\ \text{there exists } (\alpha,\beta,\gamma,m) \in \Omega'_D \text{ with } m > n \text{ and} \\ \psi(a,b,c,n;X) = \psi(\alpha,\beta,\gamma,m;X) \end{array} \right\}$$
$$\Omega^2_D(x) := \{(a,b,c,n) \in \Omega'_D : \psi(a,b,c,n;X)_2 > 0 > \psi(a,b,c,n;x)\}$$
$$\Omega^3_D(x) := \{(a,b,c,n) \in \Omega'_D : \psi(a,b,c,n;X)_2 < 0,\ \psi(a,b,c,n;x) < 0\}$$
$$\Omega^4_D(x) := \{(a,b,c,n) \in \Omega'_D : \psi(a,b,c,n;X)_2 > 0,\ \psi(a,b,c,n;x) > 0\}$$
$$\Omega^5_D(x) := \{(a,b,c,n) \in \Omega'_D : \psi(a,b,c,n;X)_2 = 0 \text{ or } \psi(a,b,c,n;x) = 0\}$$

and for convenience we will often drop the dependence on x. We wish to study the behavior of these sets with respect to the map $\phi : \mathbb{Z}^3 \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}^3 \times \mathbb{Z}_{>0}$ given by

$$(a,b,c,n) \mapsto (-c, -b - 2a_{n+1}c, -a - a_{n+1}b - a_{n+1}^2 c, n+1),$$

which is found by taking the coefficients of

$$-(aX^2 + bX + c)| \begin{pmatrix} 0 & 1 \\ 1 & a_{n+1} \end{pmatrix}.$$

We first state the following straightforward lemma, whose proof we leave to the reader.

Lemma 2.5. *The map ϕ satisfies the following:*
(a) if $(\alpha,\beta,\gamma,n+1) = \phi(a,b,c,n)$, then $b^2 - 4ac = \beta^2 - 4\alpha\gamma$,
(b) ϕ is bijective with inverse given by

$$(a,b,c,n) \mapsto (-a_n^2 a + a_n b - c, 2a_n a - b, -a, n-1),$$

(c) and $\psi(a,b,c,n;X) = -\psi(\phi(a,b,c,n);X)$.

Now, we give the following lemma, which describes the behavior of the sets Ω^i_D with respect to the map ϕ.

Lemma 2.6. *We have that*
(a) $\phi : \Omega^3_D \to \Omega^4_D$ is a bijection, and
(b) $\phi : \Omega^1_D \to \Omega^2_D$ is a bijection.

Proof. First we prove (a). Let us consider the map $\phi : \Omega^3_D \to \Omega^4_D$. We need only show that ϕ maps Ω^3_D into Ω^4_D, and that the map ϕ^{-1} given above maps Ω^4_D into Ω^3_D.

Suppose that $(a,b,c,n) \in \Omega^3_D$. To establish that $\phi(a,b,c,n) \in \Omega^4_D$, (by Lemma 2.5(a)) we need only check that

$$\begin{aligned} -c &< 0 \\ -a - a_{n+1}b - a_{n+1}^2 c &> 0 \\ \psi(\phi(a,b,c,n);X)_2 &> 0 \\ \psi(\phi(a,b,c,n);x) &> 0. \end{aligned}$$

Note that the first inequality is clear since $c > 0$, and the third and fourth inequalities hold by Lemma 2.5(c) since $(a,b,c,n) \in \Omega_D^3$. To establish the second inequality, note that

$$\psi(\phi(a,b,c,n);x) = -c\delta_{n+2}^2 + (-b - 2a_{n+1}c)\delta_{n+2}\delta_{n+1} + (-a - a_{n+1}b - a_{n+1}^2c)\delta_{n+1}^2 > 0$$

and thus

$$-a - a_{n+1}b - a_{n+1}^2c > \frac{\delta_{n+2}}{\delta_{n+1}}\left(c\frac{\delta_{n+2}}{\delta_{n+1}} + b + 2a_{n+1}c\right).$$

Thus we have the desired inequality if $b \geq -c\frac{\delta_{n+2}}{\delta_{n+1}} - 2a_{n+1}c$. If not,

$$-b > c\frac{\delta_{n+2}}{\delta_{n+1}} + 2a_{n+1}c,$$

so

$$-a - a_{n+1}b - a_{n+1}^2c > -a + a_{n+1}c\frac{\delta_{n+2}}{\delta_{n+1}} + a_{n+1}^2c > 0$$

as desired.

Now, suppose that $(a,b,c,n) \in \Omega_D^4$. Notice that $n \geq 1$, since if $n = 0$, we would have $\psi(a,b,c,0;X)_2 = (aX^2 + bX + c)_2 = a < 0$. Thus $\phi^{-1}(a,b,c,n)$ is defined, and one can show that $\phi^{-1}(a,b,c,n) \in \Omega_D^3$ by a similar argument as above. This completes the proof of (a).

In order to establish (b), let us consider the map $\phi : \Omega_D^1 \to \Omega_D^2$. As above, we need only show that ϕ maps Ω_D^1 into Ω_D^2, and that ϕ^{-1} maps Ω_D^2 into Ω_D^1.

First suppose that $(a,b,c,n) \in \Omega_D^2$. As above, one can check that both $\phi(a,b,c,n) \in \Omega_D^1 \cup \Omega_D^0$ and $\phi^{-1}(a,b,c,n) \in \Omega_D^1 \cup \Omega_D^0$. Thus it follows that $\phi^{-1}(a,b,c,n) \in \Omega_D^1$, as desired.

Now suppose that $(a,b,c,n) \in \Omega_D^1$ and choose $(\alpha,\beta,\gamma,m) \in \Omega_D'$ with m minimal such that $m > n$ and $\psi(a,b,c,n;X) = \psi(\alpha,\beta,\gamma,m;X)$. As before, to check that $\phi(a,b,c,n) \in \Omega_D^2$, we need only show that $-a - a_{n+1}b - a_{n+1}^2c > 0$.

First, consider the case where $m = n+1$. Then we have that

$$\begin{aligned}\psi(\alpha,\beta,\gamma,n+1;X) &= \psi(a,b,c,n;X) = -\psi(\phi(a,b,c,n);X)\\ &= \psi(c, b + 2a_{n+1}c, a + a_{n+1}b + a_{n+1}^2c, n+1),\end{aligned}$$

so by Lemma 2.4 we have that $(\alpha,\beta,\gamma) = (c, b + 2a_{n+1}c, a + a_{n+1}b + a_{n+1}^2c) \notin \Omega_D'$, which is a contradiction, so we cannot have $m = n+1$.

Now, suppose that $m = n+2$. Since $\psi(a,b,c,n;X) = \psi(\alpha,\beta,\gamma,m;X)$, it follows by Lemma 2.4 that $\phi(a,b,c,n) = \phi^{-1}(\alpha,\beta,\gamma,m)$, and thus

$$(-c, -b - 2ca_{n+1}, -a - ba_{n+1} - ca_{n+1}^2) = (-\alpha a_m^2 + \beta a_m - \gamma, 2\alpha a_m - b, -\alpha).$$

Thus we have that $-a - a_{n+1}b - a_{n+1}^2c > 0$ as desired.

Finally, consider the case where $m > n+2$, and here assume for the sake of contradiction that $-a - a_{n+1}b - a_{n+1}^2c \leq 0$. By minimality of m, note that $\phi^{-1}(\alpha,\beta,\gamma,m) \notin \Omega_D^2$, so it follows that

$$-\alpha a_m^2 + \beta a_m - \gamma \geq 0.$$

Since $\psi(\phi(a,b,c,n);X) = \psi(\phi^{-1}(\alpha,\beta,\gamma,m);X)$, we have

$$\begin{aligned} c\delta_{n+2}^2 + (a + ba_{n+1} + ca_{n+1}^2)\delta_{n+1}^2 + (b + 2ca_{n+1})\delta_{n+1}\delta_{n+2} \\ =(\alpha a_m^2 - \beta a_m + \gamma)\delta_m^2 + \alpha\delta_{m-1}^2 + (\beta - 2\alpha a_m)\delta_{m-1}\delta_m \\ cq_{n+1}^2 + (a + ba_{n+1} + ca_{n+1}^2)q_n^2 - (b + 2ca_{n+1})q_n q_{n+1} \\ =(\alpha a_m^2 - \beta a_m + \gamma)q_{m-1}^2 + \alpha q_{m-2}^2 - (\beta - 2\alpha a_m)q_{m-2}q_{m-1} \end{aligned}$$

and thus we have

$$\begin{aligned} (\beta - 2\alpha a_m)\delta_{m-1}\delta_m - (b + 2ca_{n+1})\delta_{n+1}\delta_{n+2} \\ = c\delta_{n+2}^2 + (a + ba_{n+1} + ca_{n+1}^2)\delta_{n+1}^2 - (\alpha a_m^2 - \beta a_m + \gamma)\delta_m^2 - \alpha\delta_{m-1}^2 \geq 0 \\ (\beta - 2\alpha a_m)q_{m-2}q_{m-1} - (b + 2ca_{n+1})q_n q_{n+1} \\ = -cq_{n+1}^2 - (a + ba_{n+1} + ca_{n+1}^2)q_n^2 + (\alpha a_m^2 - \beta a_m + \gamma)q_{m-1}^2 + \alpha q_{m-2}^2 \leq 0. \end{aligned}$$

Together, these give

$$\frac{\delta_{n+1}\delta_{n+2}}{\delta_{m-1}\delta_m} \leq \frac{\beta - 2\alpha a_m}{b + 2ca_{n+1}} \leq \frac{q_n q_{n+1}}{q_{m-2}q_{m-1}}.$$

This is a contradiction, since it is known that $\frac{q_n q_{n+1}}{q_{m-2}q_{m-1}} < 1 < \frac{\delta_{n+1}\delta_{n+2}}{\delta_{m-1}\delta_m}$. □

Now, we present a lemma which highlights the differences between the case where D is a square and the case where D is not a square.

Lemma 2.7. *(a) D is not a square, then*

$$\sum_{(a,b,c,n)\in\Omega_D^5} \psi(a,b,c,n;x) = 0.$$

(b) If $D = m^2$ for some positive integer m, then

$$\sum_{(a,b,c,n)\in\Omega_D^5} \psi(a,b,c,n;x) = \frac{\mathbb{B}(mx) - \overline{\mathbb{B}}(mx)}{2}.$$

(c) We have that

$$\sum_{\substack{n\geq 0 \\ 1\leq a\leq ma_{n+1}}} \psi(-a,m,0;x) = \frac{1}{2}\mathbb{B}(mx) - \frac{1}{12} + \frac{m^2}{2} - \frac{m^2}{2}\kappa(x).$$

Proof. First we consider the case where D is not a square. Let $(a,b,c,n) \in \Omega'_D$ and set $\psi(a,b,c,n;X) = \alpha X^2 + \beta X + \gamma$. One can check that $\beta^2 - 4\alpha\gamma = b^2 - 4ac = D$, so since D is not a square, we have that $\psi(a,b,c,n;X)_2 = \alpha \neq 0$. Thus

$$\sum_{(a,b,c,n)\in\Omega_D^5} \psi(a,b,c,n;x) = 0,$$

completing the proof of (a).

Now set $D = m^2$. We wish to characterize $(a,b,c,n) \in \Omega'_D$ such that $\psi(a,b,c,n;X)_2 = 0$. That is, we wish to study $(a,b,c,n) \in \Omega'_D$ with

$$b = a\frac{q_n}{q_{n-1}} + c\frac{q_{n-1}}{q_n}.$$

For such tuples, it follows that

$$\begin{aligned}\psi(a,b,c,n;X) &= \left((-1)^n(aq_n^2 - cq_{n-1}^2)/q_nq_{n-1}\right)X + (-1)^{n+1}(ap_nq_n - cp_{n-1}q_{n-1})/q_nq_{n-1} \\ &= (-1)^{n+1}(mX - n_0)\end{aligned}$$

where n_0 is a positive integer and $n \neq 0$.

For such tuples with $n = 1$, one can check that $(a,b,c,1) = (-n_0, m-2n_0a_1, ma_1-n_0a_1^2, 1)$. Here we have

$$\phi^{-1}(-n_0, m-2n_0a_1, ma_1-n_0a_1^2, 1) = (0, -m, n_0, 0) \notin \Omega'_D,$$

but if $n > 1$ we have

$$\phi^{-1}(a,b,c,n) = \left(\frac{a_naq_{n-2}}{q_{n-1}} - \frac{cq_{n-2}}{q_n}, 2a_na - \frac{aq_n}{q_{n-1}} - \frac{cq_{n-1}}{q_n}, -a, n-1\right) \in \Omega'_D,$$

Thus the 4-tuples we wish to characterize here are of the form

$$\phi^k(0,-m,n_0,0)$$

for $k \geq 1$ and $n_0 \geq 1$. We need only work to determine which choices of k and n_0 give $\phi^k(0,-m,n_0,0) \in \Omega'_D$.

In order to do this, we must better understand $\phi^k(0,-m,n_0,0)$, which is computed by iteratively applying k matrices to the polynomial $-mX+n_0$. That is, we need only find the coefficients of the polynomial

$$(-1)^k(-mX+n_0)|\begin{pmatrix}0 & 1\\ 1 & a_1\end{pmatrix}|\begin{pmatrix}0 & 1\\ 1 & a_2\end{pmatrix}|\cdots|\begin{pmatrix}0 & 1\\ 1 & a_k\end{pmatrix}.$$

Since one can prove inductively that

$$\begin{pmatrix}0 & 1\\ 1 & a_1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & a_2\end{pmatrix}\cdots\begin{pmatrix}0 & 1\\ 1 & a_k\end{pmatrix} = \begin{pmatrix}p_{k-1} & p_k\\ q_{k-1} & q_k\end{pmatrix},$$

we have that $\phi^k(0,-m,n_0,0)$ is found by taking the coefficients of the polynomial

$$(-1)^k(-mX+n_0)|\begin{pmatrix}p_{k-1} & p_k\\ q_{k-1} & q_k\end{pmatrix} = (-1)^k\left[q_{k-1}(n_0q_{k-1} - mp_{k-1})X^2 + (\cdots)X + q_k(n_0q_k - mp_k)\right].$$

That is, $\phi^k(0,-m,n_0,0) \in \Omega'_D$ if and only if

$$\begin{aligned}(-1)^k(n_0q_{k-1} - mp_{k-1}) &< 0\\ (-1)^k(n_0q_k - mp_k) &> 0,\end{aligned}$$

i.e.,

$$\frac{(-1)^kp_k}{q_k} < \frac{(-1)^kn_0}{m} < \frac{(-1)^kp_{k-1}}{q_{k-1}}.$$

Finally, since $p_k/q_k > x$ when k is odd and $p_k/q_k < x$ when k is even, and $\psi(\phi^k(0,-m,n_0,0);X) = (-1)^{k+1}(mX-n_0)$, we have that

$$\sum_{\substack{k\geq 1\\ \phi^k(0,-m,n_0,0)\in\Omega'_D}} \psi(\phi^k(0,-m,n_0,0);x) = \begin{cases} mx - n_0 & 0 < \frac{n_0}{m} < x\\ 0 & \text{otherwise}\end{cases}.$$

Then summing over $n_0 \geq 1$ gives

$$\begin{aligned}\sum_{(a,b,c,n)\in\Omega_D^5} \psi(a,b,c,n;x) &= \sum_{n_0\geq 1} \sum_{\substack{k\geq 1\\ \phi^k(0,-m,n_0,0)\in\Omega'_D}} \psi(\phi^k(0,-m,n_0,0);x)\\ &= \sum_{n_0\geq 1} \max(0, mx-n_0)\\ &= \frac{\mathbb{B}(mx)-\overline{\mathbb{B}}(mx)}{2},\end{aligned}$$

as desired (note that the last equality can be found on page 1162 of [9]).

Finally, in order to establish (c), we follow a computation in Section 10 of [9]. Define

$$\varepsilon_n := \sum_{a=1}^{ma_{n+1}} \psi(-a,m,0,n;x).$$

By rearranging as in [9], one can prove that $\varepsilon_n = \frac{m}{2}\left(m\delta_n^2 - m\delta_{n+2}^2 - \delta_n\delta_{n+1} + \delta_{n+1}\delta_{n+2}\right)$, so it follows as in [9] that

$$\sum_{\substack{n\geq 0\\ 1\leq a\leq ma_{n+1}}} \psi(-a,m,0;x) = \sum_{n=0}^{\infty} \varepsilon_n = \frac{1}{2}\mathbb{B}_2(mx) - \frac{1}{12} + \frac{m^2}{2} - \frac{m}{2}\kappa(x),$$

as desired. □

3. Proofs of Theorem 1.1 and Corollaries 1.2 and 1.3

3.1. **Proof of Theorem 1.1.** First consider the case where D is not a square. Recall from Section 2.3 that that all of the polynomials of the form $\psi(a,b,c,n;X)$, where $(a,b,c,n)\in\Omega_D^0$, are distinct and contained in Ω_D, so we need only check that there are no others. In order to do this, we need only show that

$$\sum_{(a,b,c,n)\in\Omega_D^0} \psi(a,b,c,n;x) = A_D(x).$$

To see this, recall from Section 2.3 that $\sum_{(a,b,c,n)\in\Omega'_D} \psi(a,b,c,n;x) = A_D(0)$. Thus we have

$$\begin{aligned}\sum_{(a,b,c,n)\in\Omega_D^0} \psi(a,b,c,n;x) &= \sum_{(a,b,c,n)\in\Omega_D^0} \psi(a,b,c,n;x)\\ &\quad + \left\{\sum_{(a,b,c,n)\in\Omega_D^1\cup\Omega_D^2} + \sum_{(a,b,c,n)\in\Omega_D^3\cup\Omega_D^4} + \sum_{(a,b,c,n)\in\Omega_D^5}\right\}\psi(a,b,c,n;x)\\ &= \sum_{(a,b,c,n)\in\Omega'_D} \psi(a,b,c,n;x) = A_D(0) = A_D(x)\end{aligned}$$

by Lemma 2.5(c), Lemma 2.6, and Lemma 2.7. This completes the proof of Theorem 1.1 for non-square D.

If $D = m^2$, then the proof is similar; here, the computation in Section 2.3 gives that

$$\sum_{(a,b,c,n)\in\Omega'_D} \psi(a,b,c,n;x) = A^*_{m^2}(0),$$

so we have

$$\begin{aligned}
&\sum_{(a,b,c,n)\in\Omega^0_D} \psi(a,b,c,n;x) + \sum_{\substack{n\geq 0\\ 1\leq a\leq ma_{n+1}}} \psi(-a,m,0,n;x) - \frac{1}{2}\overline{\mathbb{B}}_2(mx) + \frac{m^2}{2}\kappa(x)\\
&= \left(\sum_{(a,b,c,n)\in\Omega'_D} - \sum_{(a,b,c,n)\in\Omega^5_D}\right)\psi(a,b,c,n;x) + \left(\frac{1}{2}\mathbb{B}(mx) - \frac{1}{12} + \frac{m^2}{2} - \frac{m^2}{2}\kappa(x)\right)\\
&-\frac{1}{2}\overline{\mathbb{B}}_2(mx) + \frac{m^2}{2}\kappa(x)\\
&= \left(A^*_{m^2}(0) - \frac{\mathbb{B}(mx) - \overline{\mathbb{B}}(mx)}{2}\right) + \frac{1}{2}\mathbb{B}(mx) - \frac{1}{12} + \frac{m^2}{2} - \frac{1}{2}\overline{\mathbb{B}}_2(mx)\\
&= A^*_{m^2}(0) - \frac{1}{12} + \frac{m^2}{2} = A_{m^2}(0) = A_{m^2}(x)
\end{aligned}$$

as desired.

3.2. **Proof of Corollary 1.2.** By Lemma 2.3, we need only show that $\#\Omega_D(x) = +\infty$ if x is quadratic over $\mathbb{Q}$. Without loss of generality, suppose that $0 < x < 1$.

It is known [2] that there is at least one binary quadratic form $aX^2 + bXY + cY^2$ of discriminant D which is *reduced*, i.e., (since D is positive)

$$0 < \frac{\sqrt{D} - b}{2|a|} < 1 < \frac{\sqrt{D} + b}{2|a|}.$$

Note that a and c have opposite signs, since if they have the same sign, we have $D = b^2 - 4ac < b^2$, so $D - b^2 < 0$, and this contradicts the fact that $0 < \sqrt{D} - b < \sqrt{D} + b$. Thus we may assume without loss of generality that $a < 0 < c$ (since either $aX^2 + bXY + cY^2$ or $-aX^2 + bXY - cY^2$ will satisfy this property).

For these reduced binary quadratic forms , we now claim that the polynomials $\psi(a,b,c,n;X)$ (for $n \geq 0$) are all distinct and contained in $\Omega_D(x)$, i.e., that $(a,b,c,n) \in \Omega^0_D$. Note that

$$\psi(a,b,c,n,X)_2 = aq_n^2 - bq_nq_{n-1} + cq_{n-1}^2 = q_{n-1}^2\left[a(q_n/q_{n-1})^2 - b(q_n/q_{n-1}) + c\right] < 0$$

since $q_n/q_{n-1} \geq 1$, and $aX^2 - bX + c < 0$ for $X \geq 1$ since $\frac{\sqrt{D}-b}{2|a|} < 1$. Similarly,

$$\psi(a,b,c,n,x) = a\delta_{n+1}^2 + b\delta_{n+1}\delta_n + c\delta_n^2 = \delta_n^2\left[a(\delta_{n+1}/\delta_n)^2 + b(\delta_{n+1}/\delta_n) + c\right] > 0$$

since $\delta_{n+1}/\delta_n \leq 1$ and $1 < \frac{\sqrt{D}+b}{2|a|}$. Thus we have that $(a,b,c,n) \in \Omega^0_D \cup \Omega^1_D$. If $(a,b,c,n) \in \Omega^1_D$, then $\psi(a,b,c,n) \in \Omega^2_D$, and in particular

$$-a - ba_{n+1} - ca_{n+1}^2 > 0.$$

This is a contradiction since $\frac{1}{a_{n+1}} \leq 1 < \frac{\sqrt{D}+b}{2|a|}$. Thus $(a,b,c,n) \in \Omega^0_D$ for all $n \geq 0$ as desired.

3.3. **Proof of Corollary 1.3.** Suppose that $D = 5$ and $0 < x < 1$. Then recall from Section 1 that

$$A_5(0) = \{-X^2 + X + 1, -X^2 - X + 1\},$$

so by Theorem 1.1 we have that

$$\Omega_D(x) \subseteq \{\psi(-1, 1, 1; X) : n \geq 0\} \cup \{\psi(-1, -1, 1; X) : n \geq 0\}.$$

Also, since $-X^2 + XY + Y^2$ is a reduced binary quadratic form, it follows from the proof of Corollary 1.2 that

$$\{\psi(-1, 1, 1; X) : n \geq 0\} \subseteq \Omega_D(x).$$

Since $\psi(-1, \pm 1, 1; x) = -\delta_{n+1}^2 \pm \delta_n \delta_{n+1} + \delta_n^2$, this proves the first statement of Corollary 1.3.

Furthermore, suppose that n is chosen such that $a_n \neq 1$ and $a_{n+1} \neq 1$. One can show that $(-1, -1, 1, n) \in \Omega_D^0$ as in the proof of Corollary 1.2, so $-\delta_{n+1}^2 - \delta_n \delta_{n+1} + \delta_n^2$ appears as a summand as desired.

4. Examples

Here we consider discriminant $D = 5$, and various choices of x. Recall that

$$\Omega_5(0) = \{-X^2 \pm X + 1\}.$$

If we first consider $x = \frac{\sqrt{5}-1}{2} = [0; 1, 1, \ldots]$, one can compute that the polynomials $\psi(-1, -1, 1, n; X)$ are given by

$$\begin{aligned}
\psi(-1, -1, 1, 0; X) &= -X^2 + X + 1 \in \Omega_5\left((\sqrt{5} - 1)/2\right)\\
\psi(-1, -1, 1, 1; X) &= -X^2 + 3X - 1 \in \Omega_5\left((\sqrt{5} - 1)/2\right)\\
\psi(-1, -1, 1, 2; X) &= -5X^2 + 5X - 1 \in \Omega_5\left((\sqrt{5} - 1)/2\right)\\
\psi(-1, -1, 1, 3; X) &= -11X^2 + 15X - 5 \in \Omega_5\left((\sqrt{5} - 1)/2\right)\\
&\vdots
\end{aligned}$$

and one has that $-\delta_{n+1}^2 + \delta_n \delta_{n+1} + \delta_n^2 = \psi(-1, -1, 1, n; x)$ appears in (1.1) for all n (as guaranteed by the proof of Corollary 1.3). The $\psi(-1, 1, 1, n; X)$ are given by

$$\begin{aligned}
\psi(-1, 1, 1, 0; X) &= -X^2 - X + 1 \notin \Omega_5\left((\sqrt{5} - 1)/2\right)\\
\psi(-1, 1, 1, 1; X) &= X^2 + X - 1 \notin \Omega_5\left((\sqrt{5} - 1)/2\right)\\
\psi(-1, 1, 1, 2; X) &= -X^2 - X + 1 \notin \Omega_5\left((\sqrt{5} - 1)/2\right)\\
\psi(-1, 1, 1, 3; X) &= X^2 + X - 1 \notin \Omega_5\left((\sqrt{5} - 1)/2\right)\\
&\vdots
\end{aligned}$$

and one can verify that *none* of the terms appearing in (1.1) are of the form $\psi(-1, 1, 1, n; x) = -\delta_{n+1}^2 - \delta_n \delta_{n+1} + \delta_n^2$.

On the other hand, for $x = \sqrt{2} - 1 = [0; 2, 2, \ldots]$, we have

$$\begin{aligned}
\psi(-1,-1,1,0;X) &= -\,X^2 + X + 1 \in \Omega_5\left((\sqrt{5}-1)/2\right)\\
\psi(-1,-1,1,1;X) &= -\,5X^2 + 5X - 1 \in \Omega_5\left((\sqrt{5}-1)/2\right)\\
\psi(-1,-1,1,2;X) &= -\,31X^2 + 25X - 5 \in \Omega_5\left((\sqrt{5}-1)/2\right)\\
\psi(-1,-1,1,3;X) &= -\,179X^2 + 149X - 31 \in \Omega_5\left((\sqrt{5}-1)/2\right)\\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
\psi(-1,1,1,0;X) &= -\,X^2 - X + 1 \in \Omega_5\left((\sqrt{5}-1)/2\right)\\
\psi(-1,1,1,1;X) &= -\,X^2 + 3X - 1 \in \Omega_5\left((\sqrt{5}-1)/2\right)\\
\psi(-1,1,1,2;X) &= -\,11X^2 + 7X - 1 \in \Omega_5\left((\sqrt{5}-1)/2\right)\\
\psi(-1,1,1,3;X) &= -\,59X^2 + 51X - 11 \in \Omega_5\left((\sqrt{5}-1)/2\right)\\
&\vdots
\end{aligned}$$

and one can show that *all* of the values $-\delta_{n+1}^2 \pm \delta_n\delta_{n+1} + \delta_n^2$ appear.

As Zagier described in [9], if $x = \frac{1}{\pi}$ then some of these values appear as summands, while others do not.

5. Proof of Theorem 1.4 and Corollary 1.5

First let us define the weight $\frac{5}{2}$ Cohen-Eisenstein series. For nonnegative integers N, we define $H(2,N)$ as follows: if $N = 0$, then set $H(2,0) := \zeta(-3)$, and if $N \equiv 2, 3 \pmod 4$, then set $H(2,N) := 0$. For a positive integer N with $Dn^2 = N^2$, where D is a fundamental discriminant, set

$$H(2,N) := L(-1,\chi_D)\sum_{d|n}\mu(d)\chi_D(d)d\sigma_3(n/d). \tag{5.1}$$

Now define the Cohen-Eisenstein series by

$$H_2(z) := \sum_{N=0}^{\infty} H(2,N)q^N = \frac{1}{120} - \frac{1}{12}q - \frac{7}{12}q^4 - \frac{2}{5}q^5 - q^8 - \frac{25}{12}q^9 - \cdots.$$

Cohen [1] proved that $H_2(z) \in M_{5/2}(\Gamma_0(4))$, and we have that

$$H_2(z) = \frac{1}{120}\left(\theta(z)^5 - 20\theta(z)F(z)\right),$$

where $\theta(z) := \sum_{n=-\infty}^{\infty} q^{n^2} \in M_{1/2}(\Gamma_0(4))$ and $F(z) := \sum_{n=0}^{\infty}\sigma_1(2n+1)q^{2n+1} \in M_2(\Gamma_0(4))$.

Now if D is a positive fundamental discriminant, note that

$$-5H(2,D) = -5L(-1,\chi_D) = A_D(x)$$

by Lemma 2.2. In fact, more is true; Zagier [9] showed that

$$-5H(2,D) = A_D(x) = \alpha_D$$

for all nonnegative integers D which are congruent to 0 or 1 modulo 4. Thus we must study

$$-5H_2(z) = -\frac{1}{24} + \frac{5}{12}q + \frac{35}{12}q^4 + 2q^5 + 5q^8 + \frac{125}{12}q^9 + \cdots = \sum_{n\geq 0} \alpha_n q^n.$$

We first note that $-5H_2(z)$ behaves well under the Hecke operator T_{p^2}. For $f(z) = \sum_{n\geq 0} a(n)q^n \in M_{5/2}(\Gamma_0(4))$ and a prime $p > 2$, the Hecke operator is defined by

$$f(z)|T_{p^2} = \sum_{n\geq 0} \left[a(p^2n) + p\left(\frac{n}{p}\right)a(n) + p^3 a(n/p^2) \right] q^n,$$

and it is known that $f(z)|T_{p^2} \in M_{5/2}(\Gamma_0(4))$. First we show that $-5H_2(z)$ is an eigenfunction of the Hecke operator T_{p^2} with eigenvalue $1 + p^3$.

Lemma 5.1. *Let $p > 2$ be prime. Then we have that*

$$-5H_2(z)|T_{p^2} = -5(1 + p^3)H_2(z).$$

Proof. First note that $\dim_{\mathbb{C}} M_{5/2}(\Gamma_0(4)) = 2$, so we need only check that

$$-5H_2(z)|T_{p^2} = \frac{-(1+p^3)}{24} + \frac{5(1+p^3)}{12}q + \cdots .$$

To see this, note that

$$-5H_2(z)|T_{p^2} = \sum_{n\geq 0} \left[a_{p^2n} + p\left(\frac{n}{p}\right)\alpha_n + p^3\alpha_{n/p^2} \right] q^n,$$

so when $n = 0$ we have

$$\alpha_0 + 0 + p^3\alpha_0 = (1 + p^3)\alpha_0 = \frac{-(1+p^3)}{24}$$

and when $n = 1$ we have

$$\alpha_{p^2} + p\alpha_1 + 0 = -5H(2,p^2) + \frac{5}{12}p = -5\zeta(-1)\sum_{d|p} \mu(d)d\sigma_3(p/d) + \frac{5}{12}p$$

$$= \frac{5}{12}\left(\sigma_3(p) - p\sigma_3(1)\right) + \frac{5}{12}p = \frac{5(1+p^3)}{12}$$

as desired. ∎

Before we prove Theorem 1.4, let us first define

$$-5H_2(z)|U_p := \sum_{n\geq 0} u_p(n)q^n$$

$$-5H_2(z)|V_p := \sum_{n\geq 0} v_p(n)q^n$$

where

$$u_p(n) := \alpha_{pn}$$

$$v_p(n) := \begin{cases} \alpha_{n/p} & \text{if } p|n \\ 0 & \text{otherwise} \end{cases}.$$

It is known that $-5H_2(z)|U_p, -5H_2(z)|V_p \in M_{5/2}\left(\Gamma_0(4p), \left(\frac{4p}{\cdot}\right)\right)$ [7].

We also recall a useful theorem of Sturm [8], which states that if $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ is a modular form with integral coefficients, and $a(n) \equiv 0 \pmod{\ell}$ for all

$$n \leq \frac{k}{12}[\Gamma_0(1) : \Gamma_0(N)],$$

then $f(z) \equiv 0 \pmod{\ell}$.

Proof of Theorem 1.4. First note that since $p \equiv -1 \pmod{\ell}$, Lemma 5.1 implies that

$$\alpha_{p^2n} \equiv \left(\frac{n}{p}\right)\alpha_n + \alpha_{n/p^2} \pmod{\ell}$$

for all $n \geq 0$.

Note also that $u_p(0) = v_p(0)$, and

$$u_p(p) = \alpha_{p^2} \equiv \alpha_1 = v_p(p) \pmod{\ell},$$

but $-5H_2(z)|U_p \not\equiv -5H_2(z)V_p \pmod{\ell}$. To see this, note that since $p \equiv 2, 3 \pmod 5$, we have

$$v_p(5p^3) = \alpha_{5p^2} \equiv \left(\frac{5}{p}\right)\alpha_5 = -2 \pmod{\ell}$$

$$u_p(5p^3) = \alpha_{5p^4} \equiv \alpha_5 = 2 \pmod{\ell},$$

so $v_p(5p^3) \not\equiv u_p(5p^3) \pmod{\ell}$.

Since the Sturm bound for $-5H_2(z)|U_p + 5H_2(z)|V_p \in M_{5/2}\left(\Gamma_0(4p), \left(\frac{4p}{\cdot}\right)\right)$ is

$$\frac{5/2}{12}[\Gamma_0(1) : \Gamma_0(4p)] = \frac{5}{24} \cdot 4p \cdot \frac{3}{2} \cdot \frac{p+1}{p} = \frac{5}{4}(p+1) < 2p,$$

it follows that there exists some number $1 \leq n_p \leq \frac{5}{4}(p+1)$, $n_0 \neq p$ such that

$$u_p(n_p) \not\equiv v_p(n_p) = 0 \pmod{\ell}.$$

This completes the proof since $A_{pn_p}(x) = u_p(n_p)$. □

Proof of Corollary 1.5. Let $p_1, p_2, \ldots$ denote the primes (in increasing order) which satisfy the congruence conditions for p in Theorem 1.4. If $i < j < k$ and $p_i n_{p_i} = p_j n_{p_j} = p_k n_{p_k} =: D$ in the notation of Theorem 1.4, it follows that $p_i p_j p_k | D$, so

$$p_i p_j p_k \leq \frac{5}{4} p_i p_{i+1},$$

which is a contradiction. Thus at least half of the primes $p_1, p_2, \ldots$ result in distinct values $n_p p$ as described in Theorem 1.4.

The result then follows by Dirichlet's Theorem on primes in arithmetic progressions, since the primes $p_1, p_2, \ldots$ constitute two arithmetic progressions modulo 5ℓ, and for each such p we have

$$n_p p \leq \frac{5}{4} p(p+1).$$

□

Note that Corollary 1.5 only gives a lower bound for the number of nonzero coefficents of $-5H_2(z)$ modulo ℓ, which is not expected to be sharp (and does not even prove that a nonzero proportion of the coefficients are nonzero modulo ℓ). Naively, one might expect the proportion of nonzero coefficients mod ℓ to be $\frac{\ell-1}{2\ell}$, since half of the coefficients are 0, and we might guess that the other half are distributed evenly among the congruence classes mod ℓ. We give these expected proportions for the primes $\ell = 5, 7, 13$ in the table below.

ℓ	7	11	13
$\frac{\ell-1}{2\ell}$	0.4286	0.4545	0.4615

This guess, however, seems to be a bit higher than numerics suggest. To see this, we chose various primes $\ell > 5$ and computed the proportion

$$\delta(\ell, X) := \frac{\#\{0 < D \equiv 0, 1 \pmod 4 \leq X \ : \ \ell \nmid A_D(x)\}}{X}$$

for various large X. The following table lists the results.

X	$\delta(7, X)$	$\delta(11, X)$	$\delta(13, X)$
10^2	0.42	0.43	0.49
10^3	0.382	0.421	0.462
10^4	0.3767	0.4118	0.4485
10^5	0.37427	0.40910	0.44696

This table suggests that the values of the coefficients are not evenly distributed among the congruence classes modulo ℓ. Presumably, these numbers follow a distribution analogous to the Cohen-Lenstra distribution which is predicted for class numbers of imaginary quadratic fields.

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Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322

E-mail address: mjames7@emory.edu