Quadratic Polynomials, Period Polynomials, and Hecke Operators

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Abstract For any non-square $1 < D \equiv 0, 1 \pmod 4$, Zagier [8] defined

$$F_k(D;x) := \sum_{\substack{a,b,c\in\mathbb{Z}, a<0,\\ b^2-4ac=D}} \max(0, (ax^2+bx+c)^{k-1}).$$

Here we use the theory of periods to give identities and congruences which relate various values of $F_k(D;x)$.

1 Introduction and Statement of Results

For non-square $D \equiv 0, 1 \pmod 4$ and positive even integer k, define a function $F_k(D;x)$ as follows: For $x \in \mathbb{R}$, consider the set of polynomials $aX^2 + bX + c$ with integer coefficients and discriminant D such that $a < 0 < ax^2 + bx + c$. For each such polynomial, compute $(ax^2+bx+c)^{k-1}$ and then add the resulting values. That is, set

$$F_k(D;x) := \sum_{\substack{a,b,c\in\mathbb{Z}, a<0,\\ b^2-4ac=D}} \max(0, (ax^2+bx+c)^{k-1})$$

(and note that $F_k(D;x)$ can be defined similarly for square D using Bernoulli polynomials, although we will not consider such D here). This function has been studied thoroughly, and much is known about it (for example, Zagier proved [8] that if $k = 2$ or 4 and D fixed, then $F_k(D;x)$ is constant in x). Here, we will give additional identities which give information about relationships between values of $F_k(D;x)$ for various related values of x.

Let us begin with an example. We define an auxiliary function $F_k(D,2;x)$ by

$$\begin{aligned} F_k(D,2;x) := &-2^{10}F_k\left(D;\frac{x}{2}\right) + x^{10}F_k\left(D;\frac{2}{x}\right)\\ &-2^{10}F_k\left(D;\frac{x+1}{2}\right) + (x+1)^{10}F_k\left(D;\frac{2}{x+1}\right)\\ &-F_k\left(D;2x\right) + (2x)^{10}F_k\left(D;\frac{1}{2x}\right)\\ &-(x+1)^{10}F_k\left(D;\frac{2x}{x+1}\right) + (2x)^{10}F_k\left(D;\frac{x+1}{2x}\right). \end{aligned}$$

Since $F_k(D;x)$ is constant for $k = 2, 4$, we will now choose $k = 6$, and also set $D = 5$.

One can compute that for $x = 3$, we have $F_6(5;3) = 2$, since the 2 polynomials $[a, b, c]$ of interest here are

$$[-1, 5, -5], \text{ and } [-1, 7, -11].$$

One also has that $F_6(5;1/3) = 18242/6561$, and that

$$\begin{aligned} F_6(5, 2; 3) :=& - 2^{10} F_6(5; 3/2) + 3^{10} F_6(5; 2/3) \\ & - 2^{10} F_6(5; 2) + 4^{10} F_6(5; 1/2) \\ & - F_6(5; 6) + 6^{10} F_6(5; 1/6) \\ & - 4^{10} F_6(5; 3/2) + 6^{10} F_6(5; 2/3) \\ =& \, 304644624. \end{aligned}$$

Thus altogether we have

$$\frac{\frac{1742}{691}(2^{11}+1)(3^{10}-1) - F_6(5,2;3)}{\frac{1742}{691}(3^{10}-1) + F_6(5;3) - 3^{10}F_6(5;1/3)} = \frac{254016000/691}{-10584000/691} = -24.$$

We now go through the same computation using a different value for x. If we choose $x = 2/7$, we have that

$$\begin{aligned} F_6(5; 2/7) &= \frac{743556578}{282475249} \\ F_6(5; 7/2) &= \frac{391}{128} \\ F_6(5, 2; 2/7) &= -\frac{1458365017050}{282475249} \end{aligned}$$

and thus we have

$$\frac{\frac{1742}{691}(2^{11}+1)((2/7)^{10}-1) - F_6(5,2;2/7)}{\frac{1742}{691}((2/7)^{10}-1) + F_6(5;2/7) - (2/7)^{10}F_6(5;7/2)} = \frac{-521408016000/195190397059}{21725334000/195190397059} = -24.$$

From these two examples, one might wonder if

$$\frac{1742}{691}(2^{11}+1)(x^{10}-1) - F_6(5,2;x) = -24\left[\frac{1742}{691}(x^{10}-1) + F_6(5;x) - x^{10}F_6(5;1/x)\right]$$

for all real numbers x. In fact, this is true, and more generally we have that

$$\frac{1742}{691}\sigma_{11}(n)(x^{10}-1) - F_6(5,n;x) = \tau(n)\left[\frac{1742}{691}(x^{10}-1) + F_6(5;x) - x^{10}F_6(5;1/x)\right] \quad (1)$$

for all $x \in \mathbb{R}$ and $n > 1$. Here, $F_k(D, n; x)$ is defined in Section 2.3 (and is similar in shape to $F_k(D, 2; x)$ defined above), $\sigma_{11}(n) = \sum_{d|n} d^{11}$, and $\tau(n)$ is a value of Ramanujan's tau-function. A similar statement holds true for other values of D as well.

For other values of k, we are not always so lucky. For example, let us consider the case where $k = 12$. For $D = 5$ and $n = 2$, one might hope that

$$\frac{1590572822}{236364091}(2^{23}+1)(x^{22}-1)-F_{12}(5,2;x) = C\left[\frac{1590572822}{236364091}(x^{22}-1)+F_{12}(5;x)-x^{22}F_{12}(5;1/x)\right]$$

for some constant C which does not depend on x. Unfortunately, this is not the case, but we do have that

$$\frac{1590572822}{236364091}(2^{23}+1)(x^{22}-1)-F_{12}(5,2;x) \equiv 0 \pmod{72}.$$

In order to explain these identities (and many others), we make use of the connection between $F_k(D;x)$ and the theory of modular forms. It is known that

$$\frac{\zeta_D(1-k)}{2\zeta(1-2k)}(x^{2k-2}-1)+F_k(D;x)-x^{2k-2}F_k(D;1/x)$$

is the "even" part of the period polynomial of a cusp form $f_k(D;z)$ of weight $2k$ (see Section 2.3). We make use of this fact to give the following theorem, which implies the above claims for $k = 6$, since S_{12} has dimension 1 and is spanned by the eigenform

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n.$$

Theorem 1. *Suppose that k is a positive even integer, $0 < D \equiv 0,1$ (mod 4) is not a square, and $n > 1$ is an integer such that $f_k(D;z)$ is an eigenform of the Hecke operator with eigenvalue λ_n. Then we have that*

$$\frac{\zeta_D(1-k)}{2\zeta(1-2k)}\sigma_{2k-1}(n)(x^{2k-2}-1)-F_k(D,n;x) = \lambda_n\left[\frac{\zeta_D(1-k)}{2\zeta(1-2k)}(x^{2k-2}-1)+F_k(D;x)-x^{2k-2}F_k\left(D;\frac{1}{x}\right)\right].$$

While unfortunately $f_k(D;z)$ is not an eigenform in general, we can use congruences to give results analogous to Theorem 1, as in the above example with $k = 12$. The following theorems are derived from congruence results of Serre and Tate from the theory of modular forms. They correspond to the case where the appropriate Hecke eigenvalues vanish modulo some value M (which simplifies the resulting formulae considerably). Note here that one cannot ever expect these Hecke eigenvalues to be equal to 0, but Theorem 3 asserts that they are *almost always* 0 mod M.

Theorem 2. *Suppose that k is a positive even integer and $0 < D \equiv 0,1$ (mod 4) is not a square. Let K and α be as described in Section 3.2, and let λ be a prime of K lying above 2. Set $e \geq 0$ such that $\lambda^e || \alpha$. Then there is a nonnegative integer c such that for every $t \geq 1$ we have that*

$$F_k(D,n;x) \equiv \frac{\zeta_D(1-k)}{2\zeta(1-2k)}\sigma_{2k-1}(n)(x^{2k-2}-1) \pmod{\lambda^{t-e}}$$

for all real numbers x and positive integers n with at least $c+t$ distinct odd prime factors.

Theorem 3. *Suppose that k is a positive even integer, let K and α be as described in Section 3.2, and let $\mathfrak{m} \subset \mathcal{O}_K$ be an ideal of norm M which is relatively prime to α. Then a positive proportion of the primes $p \equiv -1$ (mod M) have the property that*

$$F_k(D, p; x) \equiv \frac{\zeta_D(1-k)}{2\zeta(1-2k)} \sigma_{2k-1}(p)(x^{2k-2} - 1) \pmod{M}$$

for all real numbers x and non-square $0 < D \equiv 0, 1$ (mod 4). Furthermore, for almost all positive integers n, we have that

$$F_k(D, n; x) \equiv \frac{\zeta_D(1-k)}{2\zeta(1-2k)} \sigma_{2k-1}(n)(x^{2k-2} - 1) \pmod{M}$$

for all real numbers x and non-square $0 < D \equiv 0, 1$ (mod 4).

In Section 2, we will recall the necessary background material regarding period polynomials, Hecke operators, and the connection between $F_k(D; x)$ and the theory of modular forms. In Section 3, we will prove Theorems 1, 2, and 3.

2 Preliminaries

2.1 Background on Period Polynomials and Hecke Operators

First we review the theory of periods, as described in [3]. Given a cusp form $f(z) = \sum_{n \geq 0} a(n) q^n$ (where $q := e^{2\pi i z}$) of weight $2k$ on $\mathrm{SL}_2(\mathbb{Z})$, we define the *period polynomial* of f by

$$r_f(x) := \int_0^{i\infty} f(z)(x - z)^{2k-2} dz$$

and also let r_f^+ and r_f^- denote the even and odd parts of r_f, respectively.

Let $\mathbf{V} = \mathbf{V}_{2k-2}$ be the set of polynomials of degree at most $2k - 2$, and define the slash operator by

$$P|\gamma = (cx + d)^{2k-2} P\left(\frac{ax + b}{cx + d}\right)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $P \in \mathbf{V}$. One can check that $r_f(z) \in \mathbf{W}$, where

$$\mathbf{W} = \mathbf{W}_{2k-2} := \{P \in \mathbf{V} : P|(1 + S) = P|(1 + U + U^2) = 0\},$$

Here, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. We also set $\mathbf{W}^+$ and $\mathbf{W}^-$ to be the subspaces of even and odd polynomials. Finally, set $\mathbf{W}_0^+$ to be the subspace of codimension 1 of $\mathbf{W}^+$ which does not contain the polynomial $x^{2k-2} - 1$.

It is known (due to Eichler and Shimura) that the maps

$$r^+ : S_{2k} \to \mathbf{W}_0^+$$
$$r^- : S_{2k} \to \mathbf{W}^-$$

are isomorphisms.

We now wish to establish a relationship between the theory of Hecke operators and period polynomials. We recall a result of Zagier, which generalizes a result of Manin and gives the action of Hecke operators on period polynomials in a way which respects the Eichler-Shimura isomorphisms. Zagier proved [9] that if f is a cusp form of weight $2k$ on $\mathrm{SL}_2(\mathbb{Z})$ and n is a positive integer, then

$$r_{f|T_n}(x) = \sum (cx+d)^{2k-2} r_f \left(\frac{ax+b}{cx+d} \right),$$

where the sum is over matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant n satisfying

$$a > |c|, \qquad d > |b|, \qquad bc \le 0, \qquad b = 0 \Rightarrow -\frac{a}{2} < c \le \frac{a}{2}, \qquad c = 0 \Rightarrow -\frac{d}{2} < b \le \frac{d}{2} \tag{2}$$

Thus we define the Hecke operator $\tilde{T}_n$ for period polynomials by

$$r_f(x)|\tilde{T}_n := \sum (cx+d)^{2k-2} r_f \left(\frac{ax+b}{cx+d} \right) = \sum r_f|M$$

where the sum is over matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant n which satisfy (2), and note that the result of Zagier may be written as $r_{f|T_n} = r_f|\tilde{T}_n$ for all cusp forms f. One can also check that

$$(x^{2k-2} - 1)|\tilde{T}_n = \sigma_{2k-1}(n)(x^{2k-2} - 1).$$

2.2 Congruence Results from the Theory of Modular Forms

When considering cusp forms $f \in S_k$, one might be interested in forms which are eigenforms of the Hecke operator, i.e., which satisfy $f|T_n = \lambda_n f$ for some constant λ_n. While this is not always the case, it is known that analogous statements can be made in many situations using congruences.

For example, it is known [7], [5] that the action of Hecke algebras on spaces of modular forms modulo 2 is locally nilpotent, as stated in the following lemma.

Lemma 1. *Suppose that $f(z) \in M_k \cap \mathbb{Z}[[q]]$. Then there exists a positive integer i such that*

$$f(z)|T_{p_1}|T_{p_2}| \cdots |T_{p_i} \equiv 0 \pmod{2},$$

for every collection of odd primes $p_1, p_2, \ldots p_i$.

Thus for a modular form $f(z) \in M_k \cap \mathbb{Z}[[q]$ which is not congruent to 0 (mod 2), we may define its *degree of nilpotency* to be the smallest such i, i.e., there exist odd primes $\ell_1, \ell_2, \ldots \ell_{i-1}$ for which

$$f(z)|T_{\ell_1}|T_{\ell_2}|\cdots|T_{\ell_{i-1}} \not\equiv 0 \pmod{2},$$

and for every collection of odd primes $p_1, p_2, \ldots p_i$, we have that

$$f(z)|T_{p_1}|T_{p_2}|\cdots|T_{p_i} \equiv 0 \pmod{2}.$$

More generally, one might ask about modular forms which do not have integral coefficients (e.g., in the next section, we will consider modular forms with coefficients in the ring of integers of a number field). We have the following result, which also follows from the work of Tate [7].

Lemma 2. *Let k be a positive even integer and suppose that K is a number field containing the coefficients of all the weight k normalized eigenforms in S_k. Let λ be a prime of K lying above 2. Then there is an integer $c \geq 0$ such that for every $f(z) \in S_k$ with coefficients in $\mathcal{O}_{K,\lambda}$ and every $t \geq 1$ we have that*

$$f(z)|T_{p_1}|T_{p_2}|\cdots|T_{p_{c+t}} \equiv 0 \pmod{\lambda^t}$$

for all odd primes $p_1, p_2, \ldots p_{c+t}$.

One might next ask whether one can give results with a different modulus. In order to do so, we state the following lemma of Serre [6], which he proved in more generality using the theory of Galois representations and the Chebotarev Density Theorem.

Lemma 3. *Let A denote the subset of integer weight modular forms in M_k whose Fourier coefficients are in $\mathcal{O}_K$, the ring of algebraic integers in a number field K. If $\mathfrak{m} \subset \mathcal{O}_K$ is an ideal of norm M, then a positive proportion of the primes $p \equiv -1$ (mod M) have the property that*

$$f(z)|T_p \equiv 0 \pmod{\mathfrak{m}}$$

for every $f(z) \in A$.

Serre also proved the following amazing fact.

Lemma 4. *Assume the notation in Lemma 3. If $f(z) \in A$ has fourier expansion $f(z) = \sum n = 0^\infty a(n)q^n$, then there is a constant $\alpha > 0$ such that*

$$\#\{n \leq X : a(n) \not\equiv 0 \pmod{\mathfrak{m}} = O\left(\frac{X}{(\log X)^\alpha}\right).$$

If the modular form f in Lemma 4 is a Hecke eigenform, then this implies that almost all of its Hecke eigenvalues are 0 modulo $\mathfrak{m}$.

2.3 Zagier's $F_k(D;x)$ and its connection to the theory of modular forms

As before, for non-square $D \equiv 0, 1 \pmod 4$, and positive even integer k, we define

$$F_k(D;x) := \sum_{\substack{a,b,c\in\mathbb{Z}, a<0,\\ b^2-4ac=D}} \max(0, (ax^2+bx+c)^{k-1}).$$

This function is related to cusp forms of weight $2k$ in the following way, as described by Zagier in [8]: define the polynomial

$$P_k(D;x) := \sum_{\substack{b^2-4ac=D\\ a>0>c}} (ax^2+bx+c)^{k-1}.$$

Then one can easily see that

$$x^{2k-2}F_k(D;1/x) - F_k(D;x) = P_k(D;x).$$

For $k > 2$, we may also consider

$$f_k(D;z) := C_k D^{k-1/2} \sum_{b^2-4ac=D} \frac{1}{(az^2+bz+c)^k}$$

(where C_k is a constant which is not important here), and it is easy to see the $f_k(D;z)$ is a cusp form of weight $2k$ on $\mathrm{SL}_2(\mathbb{Z})$. In [3], it was shown that its even period function is given by

$$r^+_{f_{k,D}}(x) = \frac{\zeta_D(1-k)}{2\zeta(1-2k)}(x^{2k-2}-1) - P_k(D;x).$$

This gives that

$$F_k(D;x) = \frac{\zeta_D(1-k)}{2\zeta(1-2k)} + \sum_{n=1}^{\infty} \frac{a_{k,D}(n)}{n^{2k-1}} \cos(2\pi n x),$$

where we write $f_k(D;z) = \sum_{n\geq 1} a_{k,D}(n)q^n$. Additionally, we define

$$F_k(D,n;x) := \sum \left[F_k(D;x)|J - F_k(D;x)\right]|M = P_k(D;x)|\tilde{T}_n,$$

where the sum is over matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant n which satisfy (2), and $J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2.4 Examples for small k

In order to show that the above discussion can be made explicit, and to give some easy (known) consequences of the relationship between $F_k(D;z)$ and the theory of modular forms, we consider the cases where $k = 2, 4$, and 6. First consider the case where $k = 2$ or 4, which is considered extensively in [8]. Since there are no cusp forms of weight 4 or 8, we have that

$$0 = r^+_{f_{k,D}}(x) = \frac{\zeta_D(1-k)}{2\zeta(1-2k)}(x^{2k-2}-1) - P_k(D;x),$$

so $P_k(D;x) = \frac{\zeta_D(1-k)}{2\zeta(1-2k)}(x^{2k-2}-1)$. Thus we have that

$$P_k(D;x) = \frac{\zeta_D(1-k)}{2\zeta(1-2k)}(x^{2k-2}-1) = x^{2k-2}F_k(D;1/x) - F_k(D;x).$$

It follows that the function $F_k^0(D;x) := F_k(D;x) - \frac{\zeta_D(1-k)}{2\zeta(1-2k)}$ satisfies:

$$\begin{aligned} x^{2k-2}F_k^0(D;1/x) &= F_k^0(D;x) \\ F_k^0(D;x+1) &= F_k^0(D;x) \\ F_k^0(D;0) &= 0, \end{aligned}$$

so it follows that $F_k^0(D;x) = 0$ for all rational x (and thus, by continuity, for all x). That is, for $k \in \{2,4\}$, we have that $F_k(D;x)$ is the constant function

$$F_k(D;x) = \frac{\zeta_D(1-k)}{2\zeta(1-2k)}.$$

We now consider $F_k(D;x)$ where $k=6$. Since the space of cusp forms of weight 12 and level 1 is non-empty, we no longer have that $F_6(D;x)$ is a constant function as before. For example, when $D=5$, one can compute that

$$\begin{aligned} P_6(5;x) &= 2x^{10} + 10x^8 - 30x^6 + 30x^4 - 10x^2 - 2 \\ r^+_{f_{6,5}}(x) &= \frac{360}{691}x^{10} - 10x^8 + 30x^6 - 30x^4 + 10x^2 - \frac{360}{691}. \end{aligned}$$

Note here that since the relevant space of cusp forms S_{2k} is one-dimensional (and spanned by $\Delta(z)$) it follows that $f_6(D;z)$ is a multiple of $\Delta(z)$, and is an eigenform of the Hecke operator T_n for all n; thus we have that Theorem 1 applies whenever $k=6$.

3 Proofs

3.1 Proof of Theorem 1

Since $f_k(D,z)$ is an eigenform of the Hecke operator, we have that $f_k(D;z)|T_n = \lambda_n f_k(D;z)$. Thus

$$\begin{aligned} r^+_{f_{k,D}|T_n}(x) &= r^+_{\lambda_n f_{k,D}}(x) \\ r^+_{f_{k,D}}(x)|\tilde{T}_n &= \lambda_n r^+_{f_{k,D}}(x) \\ \tfrac{\zeta_D(1-k)}{2\zeta(1-2k)}\sigma_{2k-1}(n)(x^{2k-2}-1) - F_k(D,n;x) &= \lambda_n \left[\tfrac{\zeta_D(1-k)}{2\zeta(1-2k)}(x^{2k-2}-1) + F_k(D;x) - x^{2k-2}F_k\left(D;\tfrac{1}{x}\right)\right] \end{aligned}$$

as desired.

3.2 Congruences for period polynomials of modular forms

One must be a bit careful when applying the congruence results of Section 2.2; they do not necessarily apply to the cusp forms $f_k(D;z)$. Here we consider a basis of eigenforms for S_{2k} in order to circumvent this issue.

Fix a positive even integer k and a positive non-square integer $D \equiv 0, 1 \pmod 4$. Set $d_k := \dim(S_{2k})$ and let

$$f_1, f_2, \cdots, f_{d_k}$$

be a basis of eigenforms for S_{2k} which are normalized so that their corresponding even period polynomials

$$r^+_{f_1}(X), r^+_{f_2}(X), \ldots, r^+_{f_{d_k}}(X)$$

have coefficients in a number field K (where K is defined to be the smallest number field which contains all of the coefficients of the weight $2k$ normalized eigenforms of S_{2k}). Note that such a choice exists by the "The Periods Theorem" of Manin [4]. Since these eigenforms give a basis for S_{2k}, their even period polynomials give a basis for $\mathbf{W}_0^+$, so there exist constants $c_1, \ldots, c_{d_k}$ such that

$$r^+_{f_{k,D}}(X) = \sum_{i=1}^{d_k} c_i r^+_{f_i}(X).$$

Note that $r^+_{f_{k,D}}(X) \in \mathbb{Q}[X]$, and thus we have that $c_i \in K$ for all i.

Thus we may choose $\alpha \in \mathcal{O}_K$ so that

$$\alpha\left(c_i \lambda_{i,n} r^+_{f_i}(X)\right) \in \mathcal{O}_K[X]$$

for all i and $n > 1$ (where $\lambda_{i,n}$ is the eigenvalue of f_i with respect to the Hecke operator T_n). It follows that for $\mathfrak{m}$ coprime to α, and $n > 1$ such that

$$r^+_{f_i}|\tilde{T}_n \equiv 0 \pmod{\mathfrak{m}}$$

for all i, we have that

$$r^+_{f_{k,D}}|\tilde{T}_n = \sum_{i=1}^{d_k} c_i r^+_{f_i}|\tilde{T}_n \equiv 0 \pmod{\mathfrak{m}}.$$

3.3 Proof of Theorem 2

Fix a positive integer t and choose a positive integer n with at least $c + t$ distinct odd prime factors. Then by Lemma 2 we have that

$$\alpha c_i r^+_{f_i}(X)|\tilde{T}_n = \alpha c_i r^+_{f_i|T_n}(X) = \alpha c_i \lambda_{i,n} r^+_{f_i}(X) \equiv 0 \pmod{\lambda^t}$$

for all i, and thus we have that $\alpha r^+_{f_{k,D}}(X)|\tilde{T}_n \equiv 0 \pmod{\lambda^t}$. Finally, this gives

$$r^+_{f_{k,D}}(X)|\tilde{T}_n \equiv 0 \pmod{\lambda^{t-e}}$$
$$\left(\frac{\zeta_D(1-k)}{2\zeta(1-2k)}(X^{2k-2}-1) - P_k(D;X)\right)|\tilde{T}_n \equiv 0 \pmod{\lambda^{t-e}}$$
$$\sigma_{2k-1}(n)\frac{\zeta_D(1-k)}{2\zeta(1-2k)}(X^{2k-2}-1) \equiv F_k(D,n;X) \pmod{\lambda^{t-e}}$$

as desired.

3.4 Proof of Theorem 3

Note that for a positive proportion of primes $p \equiv -1 \pmod{M}$, we have that

$$\alpha c_i r^+_{f_i}(X)|\tilde{T}_p = \alpha c_i r^+_{f_i|T_p}(X) = \alpha c_i \lambda_{i,p} r^+_{f_i}(X) \equiv 0 \pmod{M}$$

for all i by Lemma 3, and thus we have that $\alpha r^+_{f_{k,D}}(X)|\tilde{T}_p \equiv 0 \pmod{M}$. Finally, this gives

$$r^+_{f_{k,D}}(X)|\tilde{T}_p \equiv 0 \pmod{M}$$
$$\left(\frac{\zeta_D(1-k)}{2\zeta(1-2k)}(X^{2k-2}-1) - P_k(D;X)\right)|\tilde{T}_p \equiv 0 \pmod{M}$$
$$\sigma_{2k-1}(p)\frac{\zeta_D(1-k)}{2\zeta(1-2k)}(X^{2k-2}-1) \equiv F_k(D,p;X) \pmod{M}.$$

This proves the first statement of Theorem 3. To see the second statement, note that Lemma 4 says that almost all positive integers n satisfy $\lambda_{i,n}$ for all i. For such n, we have that $\sigma_{2k-1}(n)\frac{\zeta_D(1-k)}{2\zeta(1-2k)}(X^{2k-2}-1) \equiv F_k(D,n;X) \pmod{M}$ by the same argument as above.

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