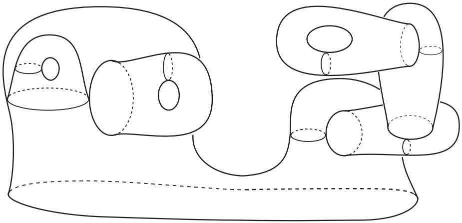
Grope concordance and a conjecture of Levine

Jim Conant, Rob Schneiderman, Peter Teichner

September 23, 2010

Motivating Question



A grope of class 4.

Question

Under what conditions do the components of a link in the 3–sphere bound class n gropes disjointly embedded in the 4–ball?

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- Conant-Teichner (2004) and Schneiderman (2005) showed that a knot bounds a grope of arbitrary class into the 4-ball provided the Arf invariant vanishes. Schneiderman gave an explicit geometric construction!

Grope Concordance Filtration

Grope concordance filtration (by class) of the set of framed links with m components $\mathbb{L}$:

$$\cdots \subseteq \mathbb{G}_3 \subseteq \mathbb{G}_2 \subseteq \mathbb{G}_1 \subseteq \mathbb{G}_0 \subseteq \mathbb{L}$$

- $\mathbb{G}_n = \mathbb{G}_n(m)$ is the set of framed links that bound class $(n+1)$ framed gropes disjointly embedded in B^4.
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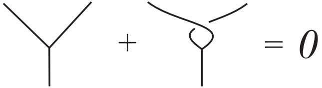
- The *associated graded* $\mathsf{G}_\mathsf{n} = \mathsf{G}_\mathsf{n}(\mathsf{m})$ is the quotient of $\mathbb{G}_n$ modulo grope concordance of class $n+2$.

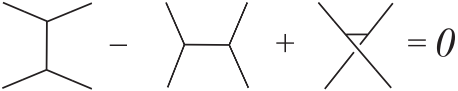
The groups $\mathscr{T}_n$

Let $\mathscr{T} = \mathscr{T}(m)$ be the free abelian group on oriented unitrivalent trees with leaves labeled by $\{1, \ldots, m\}$, modulo the antisymmetry (AS) and Jacobi (IHX) relations.

AS:



IHX:

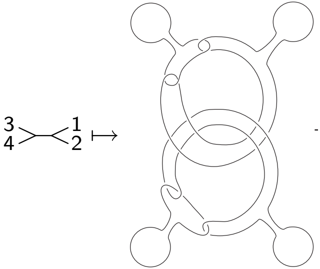


$\mathscr{T}$ inherits a grading by the number of trivalent vertices.

A construction known to at least Bing, Cochran and Habiro leads to a *realization map*

$$R_n : \mathscr{T}_n \longrightarrow \mathsf{G_n}$$

which we prove is *well-defined* and *surjective.*



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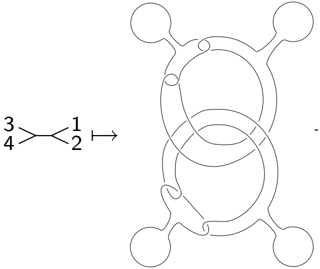

- The IHX relation in the domain $\mathscr{T}_n$ corresponds to a geometric IHX relation on embedded gropes in 4-space.

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- So $2[X, X] = 0 \in \mathsf{L}'_{2k}$. But $[X, [Y, Y]] = 0 \in \mathsf{L}'_k$ by the Jacobi identity.
- $0 \to \mathbb{Z}_2 \otimes \mathsf{L}_n \to \mathsf{L}'_{2n} \to \mathsf{L}_{2n} \to 0$ (Levine)

The groups D_n and D'_n

D_n and D'_n are defined as kernels of bracketing maps:

$$0 \to \mathsf{D}_n \to V \otimes \mathsf{L}_{n+1} \to \mathsf{L}_{n+2} \to 0$$

$$0 \to \mathsf{D}'_n \to V \otimes \mathsf{L}'_{n+1} \to \mathsf{L}'_{n+2} \to 0$$

D_n *is the natural home for* μ*-invariants.* We'll come back to this!

Corollary

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Milnor invariants of length $\leq 2n+1$ vanish on G_{2n} and the <u>total Milnor invariant</u> of length $2n+2$, μ_{2n}, fits into a commutative triangle of isomorphisms:

$$\begin{array}{ccc} \mathscr{T}_{2n} & \xrightarrow[\cong]{R_{2n}} & \mathsf{G}_{2n} \\ & \underset{\eta}{\overset{\cong}{\searrow}} & \cong \downarrow \mu_{2n} \\ & & \mathsf{D}'_{2n} \end{array}$$

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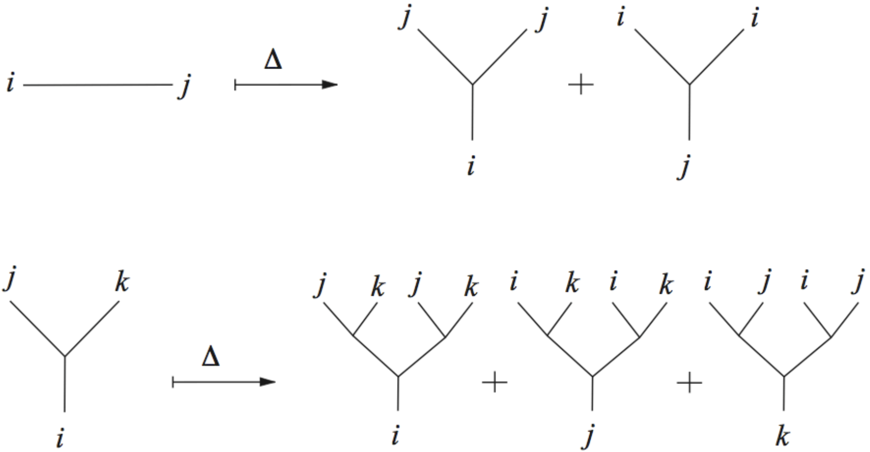
The proof of this theorem and the ones to follow uses Schneiderman's equivalence between gropes and Whitney towers.

The *reduced* version $\widetilde{\mathcal{T}}_{2n-1}$ of $\mathcal{T}_{2n-1}$ is defined by dividing out the *framing relations* (FR), which are images of homomorphisms

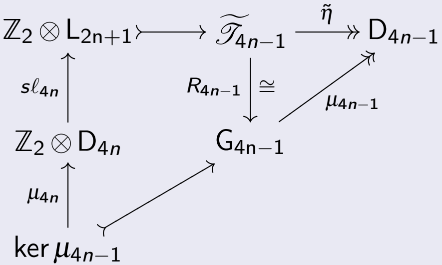
$$\Delta_{2n-1} : \mathbb{Z}/2 \otimes \mathcal{T}_{n-1} \to \mathcal{T}_{2n-1}$$

defined by sending an order $n-1$ tree t to the sum of trees gotten by doubling the subtree adjacent to each univalent vertex of t.



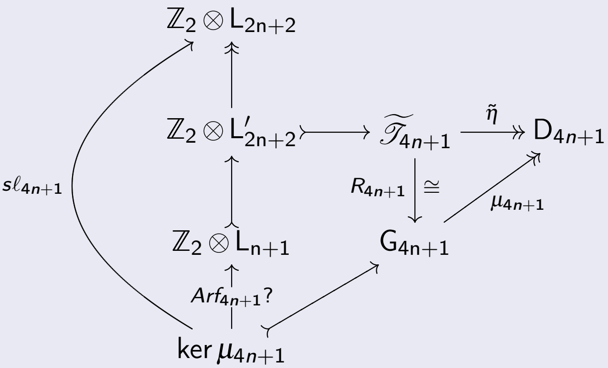
Theorem

The map R_{4n-1} is an isomorphism, detected by the total Milnor invariant of length $4n+1$ and <u>*higher-order Sato-Levine invariants.*</u>



Conjecture

$R_{4n+1} : \widetilde{\mathscr{T}}_{4n+1} \to \mathsf{G}_{4n+1}$ is an isomorphism, classified by Milnor invariants of length $4n+3$, higher-order Sato-Levine invariants, and <u>*higher order Arf invariants.*</u>



μ-invariants and D_n

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- The i-th longitude maps to the quotient $\frac{F_{n+1}}{F_{n+2}} \cong \mathsf{L}_{\mathsf{n+1}}$.

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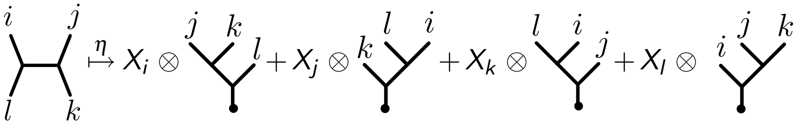
- The *first non-vanishing Milnor invariant* $\mu_n(L)$ of order n is defined to be

$$\mu_n(L) := \sum_i X_i \otimes \mu_n^i(L) \in \mathsf{D}_n \subset V \otimes \mathsf{L}_{\mathsf{n+1}}$$

Levine's map

Jerry Levine defined a homomorphism

$$\eta' \colon \mathscr{T}_n \to \mathsf{D}'_n.$$



Levine was studying the group of homology cylinders over a surface with one boundary component.

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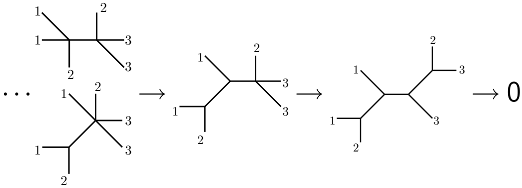
2. Levine showed that η' is onto.
3. Levine, and independently, Habiro, showed that η'_n is injective when n is prime and many other special cases.

Lifting to Chain Complexes

$\mathbb{T}_\bullet(6)$:



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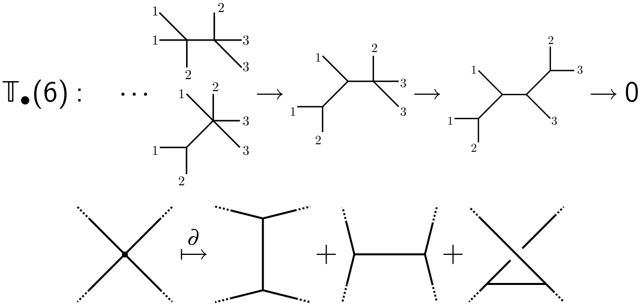

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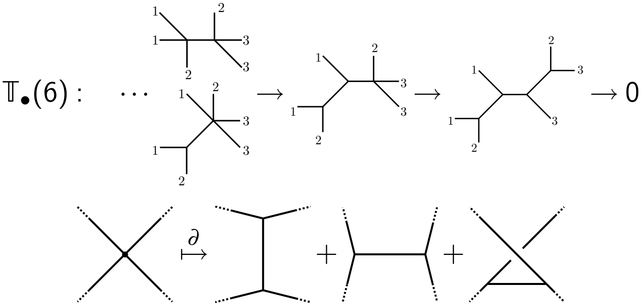
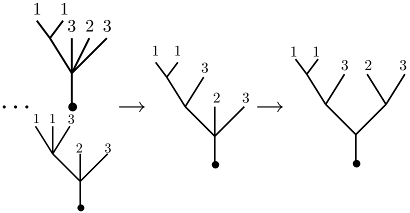


$$H_0(\mathbb{T}_\bullet(n+2)) \cong \mathcal{T}_n$$

$\mathbb{L}_\bullet(6)$:



$\longrightarrow 0$

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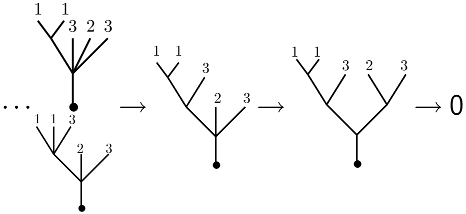

$$H_0(\mathbb{L}_\bullet(n)) \cong \mathsf{L}'_\mathsf{n}$$

Lifting the bracketing map

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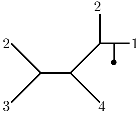

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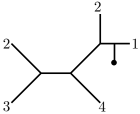

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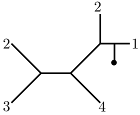

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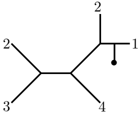

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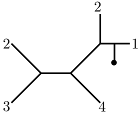

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  - $H_1(\overline{\mathbb{L}}_\bullet(n+2)) \cong H_0(\mathbb{T}_\bullet(n+2))$
- Both proofs use the technique of *discrete Morse theory* in the context of abstract chain complexes.

Definition (Homological Vector Field)

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$$\mathbf{a_1}, \mathbf{b_1}, \mathbf{a_2}, \mathbf{b_2}, \ldots, \mathbf{a_m}$$

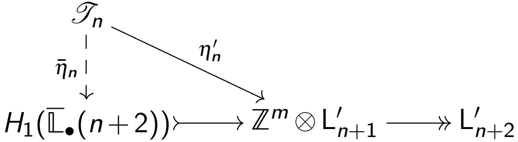
   where each $(\mathbf{a_i}, \mathbf{b_i}) \in \Delta$, and $\mathbf{a_i} \neq \mathbf{a_{i-1}}$ has nonzero coefficient in $\partial \mathbf{b_{i-1}}$.
4. *A vector field is said to be a gradient field if there are no closed gradient paths.*

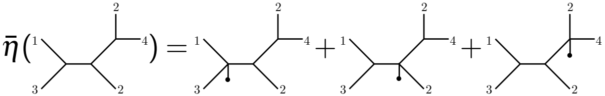
A basis element is said to be *critical* if doesn't appear in any vector of the vector field.

Theorem (Kozlov)

The chain complex (C_*, ∂) is quasi-isomorphic to a chain complex $(C_*^\Delta, \partial^\Delta)$, called the Morse complex, with basis in one-to-one correspondence with critical generators.

Key idea:
η'_n lifts uniquely to a map $\bar{\eta}_n$.

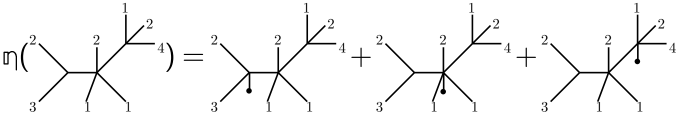




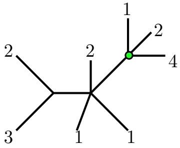
There is a *chain map*

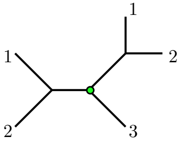
$$\boldsymbol{\eta}\colon \mathbb{T}_\bullet(n+2) \to \overline{\mathbb{L}}_{\bullet+1}(n+2)$$

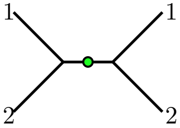
such that $H_0(\boldsymbol{\eta}) = \bar{\eta}$.



- Choose basepoints fixed by $Aut(t)$ for every $t \in \mathbb{T}_\bullet(n)$, at vertices when possible.

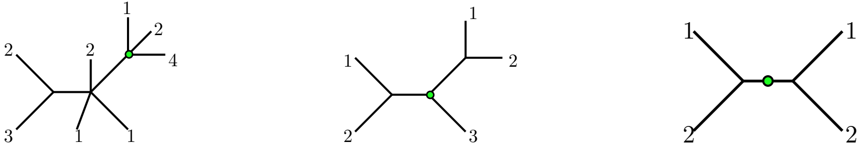




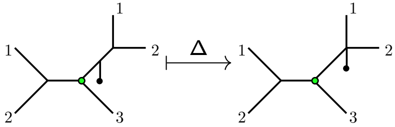


A vector field on $\overline{\mathbb{L}}_\bullet(n+2)$

- Choose basepoints fixed by $Aut(t)$ for every $t \in \mathbb{T}_\bullet(n)$, at vertices when possible.



- Define Δ by "pushing away from the basepoint."



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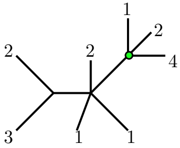

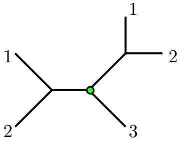

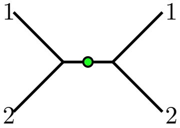

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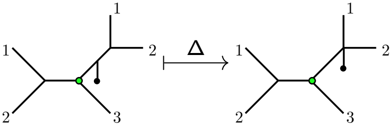
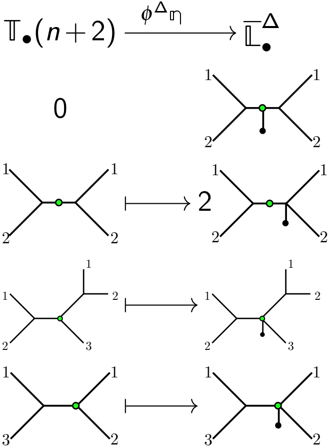
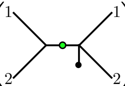
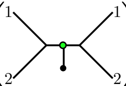


- Here we needed to generalize Kozlov's result to a special non-free case.

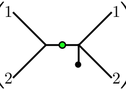
Critical Generators in $\overline{\mathbb{L}}_\bullet(n+2)$

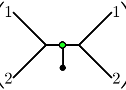


- $cok_\bullet = \bigoplus \{ 0 \to \mathbb{Z}_2 \Big($ 

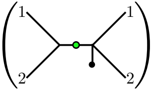
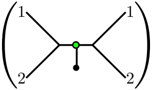
$\Big) \to \mathbb{Z}_2 \Big($ 

$\Big) \to 0 \}$

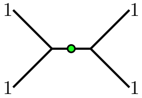
- $cok_\bullet = \bigoplus\{0 \to \mathbb{Z}_2 \Big($ 

$\Big) \to \mathbb{Z}_2 \Big($ 

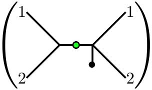
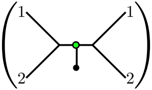
$\Big) \to 0\}$
- So the cokernel is acyclic.

- $cok_\bullet = \bigoplus \{0 \to \mathbb{Z}_2$  $\to \mathbb{Z}_2$  $\to 0\}$

- So the cokernel is acyclic.
- The kernel is not acyclic, but in degree 0, it represents 0 in homology.



$$\mapsto 0 \in H_0(\mathbb{T}_\bullet(n+2)) \cong \mathscr{T}_n$$

- $cok_{\bullet} = \bigoplus \{0 \to \mathbb{Z}_2$  $\to \mathbb{Z}_2$  $\to 0\}$

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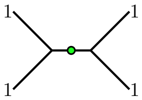

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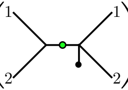
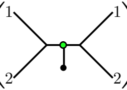
- $$cok_\bullet = \bigoplus \{0 \to \mathbb{Z}_2 \left(\right.$$ 

$$\left. \right) \to \mathbb{Z}_2 \left(\right.$$ 

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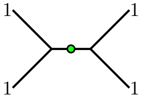

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- Thus $H_0(\eta)\colon H_0(\mathbb{T}_\bullet(n+2)) \xrightarrow{\cong} H_1(\overline{\mathbb{L}}_\bullet(n+2)) \cong \mathsf{D}'_n$
- The proof is complete!

- To show $H_1(\mathbb{L}_\bullet(n)) = 0$ we construct a vector field $\Delta = \Delta_0 \cup \Delta_1$ where $\Delta_i \colon \mathbb{L}_i \to \mathbb{L}_{i+1}$ with no critical vectors in degree 1.

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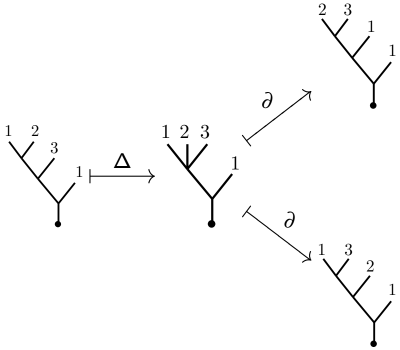
- Work first with $\mathbb{Z}[\frac{1}{2}]$ coefficients.
- $H_0(\mathbb{L}_\bullet(n); \mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}[\frac{1}{2}] \otimes \mathsf{L}_n$ has a well-known basis, called the Hall basis, so our strategy is to define $\Delta_0(J)$ to be some nontrivial contraction of J for every non-Hall tree J.



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- This is the most one can hope to do for Δ_0, because Hall trees need to survive as a basis for $H_0(\mathbb{L}_\bullet(n); \mathbb{Z}[\frac{1}{2}])$.

- All the trees in $im\Delta_0$ are not critical, so it suffices to define Δ_1 to be nonzero on all the other degree 1 trees.

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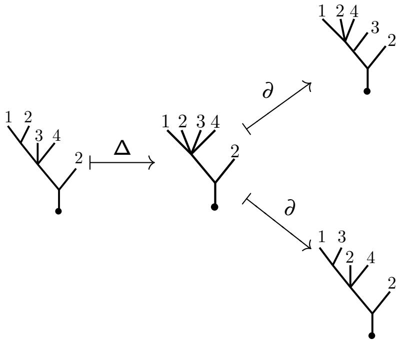
- We define Δ_1 for each different type of Hall_1 problem as a certain contraction.



- The resulting vector field is again shown to be gradient by arguing that the Hall order increases as one moves along gradient paths. This then proves the $\mathbb{Z}[\frac{1}{2}]$-coefficients case.

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- The chain complex $\mathbb{L}_\bullet(n)$ is not free, so we can't use the universal coefficient theorem, but through some trickery we are still able to conclude that $H_1(\mathbb{L}_\bullet(n); \mathbb{Z}) = 0$.