ON THE NUMBER OF TILINGS OF A SQUARE BY RECTANGLES

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ABSTRACT. We develop a recursive formula for counting the number of combinatorially distinct tilings of a square by rectangles. The resulting numbers appear to have an 8-fold periodicity modulo 2. Finally we use these results to calculate the homotopy type of some spaces of tilings.

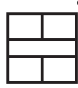
Let T_n denote the set of combinatorially distinct rectangular tilings of a square by $\leq n$ rectangles. We say that two tilings are combinatorially equivalent if there is a homeomorphism taking one to the other that fixes the four corners. Thus T_1 has 1 element, T_2 has two elements, and T_3 has six elements. We are interested in determining a formula for the numbers t_n, defined to be the number of elements in T_n.

1. RECURSIVE FORMULA FOR THE NUMBER OF TILINGS

Theorem 1. *Let $t_{m,r,e}$ be the number of distinct tilings with m tiles, r edges that meet the right-hand side of the square and e 4-valent vertices.*

$$t_{m,r,e} = \sum_{c=1}^{\lceil (s+1)/2 \rceil} \sum_{n=1}^{m-1} \sum_{s=0}^{n-1} \sum_{f=0}^{e} (-1)^{c+1} \binom{\ell-1}{c-1} \binom{s+2-\ell}{c} \binom{\ell-c}{\nabla} \binom{\Delta - c - \nabla + \ell - 1}{\ell - 1} t_{n,s,f}$$

where $\nabla = e - f$, $\Delta = m - n$, and $\ell = s + m - n - r$. The base of the recursion is given by $t_{k,k-1,0} = 1$ for $k \geq 1$.

Proof. Every rectangular tiling, except ones with only vertical edges can be generated from a simpler tiling by the process in Figure 1, where $c = 1$. The simpler tiling is pictured in (A). Then one pushes an edge of length ℓ in from the right, blocking $\ell - 1$ horizontal edges from hitting the right edge, as in (B). One then adds horizontal edges in the newly created box, some of which create 4-valent vertices as in (C), and some of which do not as in (D). However, some tilings may be generated in more than one way from this move. For example, the tiling  comes from two different simpler tilings. To take care of this we use an inclusion-exclusion argument and write

$$t_{m,r,e} = \sum_{c \geq 1} (-1)^{c+1} (\text{\# of ways to push in } c \text{ edges from the right from a simpler tiling})$$

First we count the ways to push in c lines from the right with total length ℓ, as in Figure 1 (B). (Note that $\ell = s - r + m - n$, because $\Delta\text{Boxes} = \Delta(\text{Right edges}) - \ell$.) Since there are $s + 1$ available slots on the right, this is the count of the number of c-component subsets of

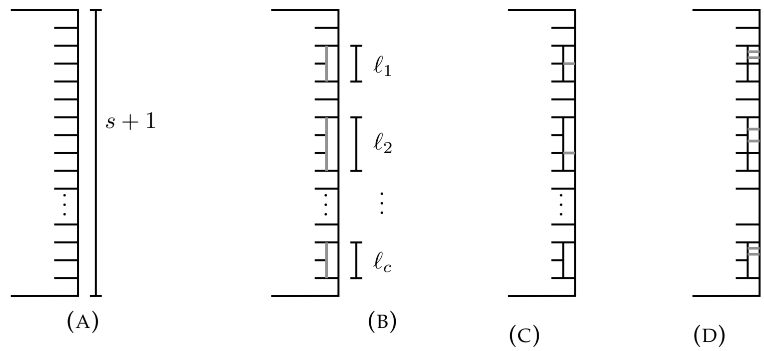


FIGURE 1. (A): The right side of the square with s edges hitting it. (B): Pushing in c vertical edges, of total length $\ell_1 + \cdots + \ell_c = \ell$. (C): Adding $e - f$ horizontal line segments to create $e - f$ 4-valent vertices. (D): Adding edges to the ℓ available bins.

$s+1$ with a total length of ℓ, which by Lemma 2, is $\binom{\ell-1}{c-1}\binom{s+2-\ell}{c}$. Next, we need to create $e - f$ 4-valent vertices, and the only way to do this is to put a horizontal line at one of the existing pushed in horizontal lines, as in (C). There are $\ell - c$ pushed in lines, so there are $\binom{\ell-c}{e-f}$ choices available. Finally, we need to figure out how to distribute the remaining horizontal edges to get an m-tile configuration with s right-hitting edges. The number of bins these new horizontal lines can go to is ℓ. Each pushed in component creates a new box making c, and each 4-valent vertex also creates a new box, making $c + e - f$. So we need to create $m - n - (c + e - f)$ new boxes. Hence we need to count the number of ways to distribute $m - n - c - e + f$ edges into the ℓ distinct slots they can go, as in (D). By Lemma 3, this is $\binom{m-n-c-e+f+\ell-1}{\ell-1}$. Thus we have accounted for all four factors of the coefficient in the formula.

The limits of the summations are explained as follows. Given a tiling where s edges hit the right edge, one can push in at most $\lceil (s+1)/2 \rceil$ edges. The number of tiles in the simpler tiling must be smaller, so n ranges to $m-1$. The number of edges meeting the right may not be smaller in the simpler tiling, but we can at least say it has to be less than the number of tiles n. Finally the number of 4-valent vertices must indeed be less than or equal to the number in the more complex tiling. $\square$

Lemma 2. *The number of c-component subsets of $\{1, \ldots, s+1\}$ of total size ℓ is given by the formula*

$$\binom{\ell-1}{c-1}\binom{s+2-\ell}{c}.$$

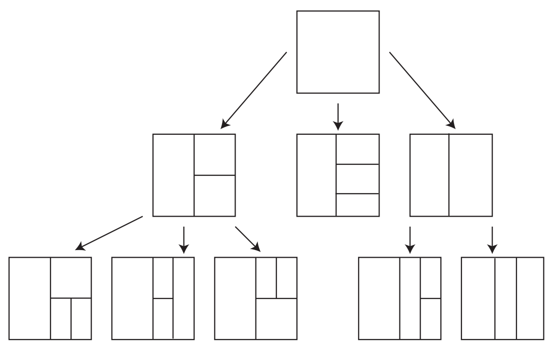
Proof. First we count the number of ways to break ℓ into c nonzero pieces, which is $\binom{\ell-1}{c-1}$. Then we count the ways of inserting those c pieces into the rest of the slots. There are

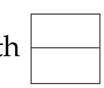
$s+1-\ell$ slots remaining, and there are $s+2-\ell$ interstices available, accounting for the $\binom{s+2-\ell}{c}$ term. □

The following lemma is well-known and can be found, for example, in [1].

Lemma 3. *The number of nonnegative integer solutions* $(x_1, \ldots, x_n)$ *to the equation* $x_1 + \cdots + x_n = m$ *is given by the formula* $\binom{m+n-1}{n-1}$.

The recursive strategy of the proof of Theorem 1 is illustrated in the following diagram. Here we start with the tiling with one rectangle, push an edge in from the right in all ways, and then again push an edge in from the right to the resulting tilings. Here we only look at tilings with ≤ 4 rectangles.



Similarly one can construct new tilings starting with  and so on.

1.1. **Calculations.** Let $t_{m,e}$ denote the number of tilings which have m rectangles and e 4-valent vertices. So $t_{m,e} = \sum_{r=0}^{m-1} t_{m,r,e}$. The above recursion yields the following data.

m	1	2	3	4	5	6	7	8	9	10	11	12
$t_{m,0}$	1	2	6	24	116	642	3938	26194	186042	1395008	10948768	89346128
$t_{m,1}$	0	0	0	1	12	114	1028	9220	83540	768916	7200852	68611560
$t_{m,2}$	0	0	0	0	0	2	48	770	10502	132210	1593934	18755516
$t_{m,3}$	0	0	0	0	0	0	0	10	348	7680	137940	2206972
$t_{m,4}$	0	0	0	0	0	0	0	0	1	104	4020	106338
$t_{m,5}$	0	0	0	0	0	0	0	0	0	0	20	1571
$t_{m,6}$	0	0	0	0	0	0	0	0	0	0	0	2
t_m	1	2	6	25	128	758	5014	36194	280433	2303918	19885534	179028087

Indeed the sequence t_m continues

$$1,2,6,25,128,758,5014,36194,280433,2303918,19885534,179028087,1671644720,\\ 16114138846,159761516110,1623972412726,16880442523007,179026930243822,\\ 1933537655138482,21231023519199575,236674460790503286,2675162663681345170,\\ 30625903703241927542,354767977792683552908,4154708768196322925749,\\ 49152046198035152483150,587011110939295781585102,7072674305834582713614923$$

2. Symmetric tiles and a mod 2 counting conjecture

The dihedral group of 8 elements D_8 acts on T_n. Let s_n be the number of tilings fixed by this action. That is, s_n counts the totally symmetric tilings.

Lemma 4. $s_n \equiv t_n \mod 2$

Proof. The orbits of the D_8 action on T_n have an even number of elements except for the singleton orbits. □

Lemma 5. *A totally symmetric tiling has either* $4k$ *tiles or* $4k+1$ *tiles.*

Proof. Since the tiling is totally symmetric D_8 acts on the rectangles within the tiling. The orbit of a tile under the D_8 action has either 1, 4, or 8 elements. It has 1 element if and only if the tile contains the square's center in its interior. □

Proposition 6. $s_n = 0$ *unless* $n = 4k$ *or* $n = 4k+1$. *Furthermore* $s_{4k+1} = s_{4k+4}$.

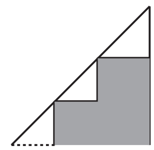
Proof. The first statement follows from Lemma 5. The bijection corresponding to $s_{4k+1} = s_{4k+4}$ is given by subdividing the central square into 4 squares. □

The sequence t_n indeed obeys these equations mod 2. This acts as a check on our work. Indeed, this sequence appears to satisfy the even stronger property that the number of tilings is even unless $n = 8k+1$ or $8k+4$ in which case it is odd. Here is t_n mod 2, for $1 \leq n \leq 28$.

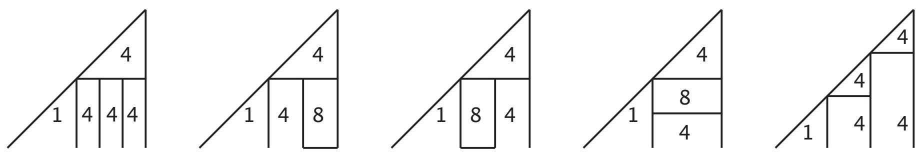
$$1,0,0,1,0,0,0,0,1,0,0,1,0,0,0,0,1,0,0,1,0,0,0,0,1,0,0,1,\ldots$$

Conjecture 7. $t_n \equiv 1 \mod 2$ *if* $n = 8k+1$ *or* $n = 8k+3$. *Otherwise* $t_n \equiv 0 \mod 2$.

Conjecture 7 can be independently verified for small n by directly counting symmetric configurations. Every symmetric tiling is determined by what it looks like in a triangular fundamental domain for the D_8 action, depicted in grey in the following picture: .
So we study the possible configurations when restricted to this triangle. It is clear that they must look as follows



where the grey region is a rectangular tiling, and there are some number of “sawteeth” that hit the diagonal. The dashed edge may or may not be there, and accounts for the equality $s_{4k+1} = s_{4k+4}$. So for example, here is a count of the symmetric tilings by 17 rectangles.



Here the numbers refer to the number of rectangles in the orbit of a given region, and must add up to 17. We see that $s_{17} = 5$, which is consistent with our calculation that $t_{17} \equiv 1 \bmod 2$.

3. Topological remarks

Let $\mathbb{T}_n$ be the topological space of tilings of the unit square by $\leq n$ rectangles. The topology is straightforward: if you move a vertex slightly the new tiling is near the old tiling. One way to make this precise is to consider the Hausdorff metric on the 1-skeleta of tilings. This makes $\mathbb{T}_n$ into a metric space.

The space $\mathbb{T}_n$ is a cell complex, where the cells correspond to combinatorially distinct tilings. The dimension of a cell with m tiles and e 4-valent vertices is $m - e - 1$.

Define the reduced Euler characteristic $\tilde{\chi}(K)$ of a complex K to be $\chi(K) - 1$. Let $x_n = \tilde{\chi}(\mathbb{T}_n)$. We can use our calculations of $t_{m,e}$ to calculate reduced Euler characteristics.

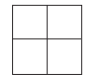
Proposition 8. *The sequence x_n of reduced Euler characteristics is given by:*

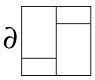
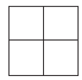
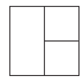
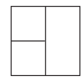
$$0, -2, 4, -19, 85, -445, 2513, -15221, 97436, -653290, 4554620, -32833261, \ldots$$

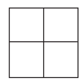
for $n \geq 1$

Proposition 9.

1. *The inclusion $\mathbb{T}_n \hookrightarrow \mathbb{T}_{n+1}$ is null-homotopic.*
2. $\mathbb{T}_1 = \{*\}$
3. $\mathbb{T}_2 \cong \vee_2 S^1$
4. $\mathbb{T}_3 \simeq \vee_4 S^2$
5. $\mathbb{T}_4 \simeq \vee_{19} S^3$
6. $\mathbb{T}_5 \simeq \vee_{85} S^4$

Proof. The inclusion $\mathbb{T}_n \hookrightarrow \mathbb{T}_{n+1}$ is null-homotopic via the deformation retraction that wipes a configuration away by expanding a thin rectangle at the bottom to fill the entire square. Thus the only possible homology is supported on the "new" cells, which are mostly in the top dimension except for cells with 4-valent vertices. In the above cases, there are no such cells until $\mathbb{T}_4$. In this case there is only one such cell, . However, in the cellular chain complex, we have

∂  $=$  $+$  $+$ ,

with appropriate choices of orientations. So  is homologous to a chain in $\mathbb{T}_3$, which is homotopically inessential in $\mathbb{T}_4$. Similarly there are 12 cells in $\mathbb{T}_5$ with a single 4-valent vertex, and only two equivalence classes under the D_8 action. By perturbing the 4-valent vertex slightly as in the $\mathbb{T}_4$ case, we see that each of these 12 cells is homologous to a chain in $\mathbb{T}_4$. Therefore, all reduced homology is supported in the top degree and one only needs to use the reduced Euler characteristic to calculate what it is. Then by the Hurewicz theorem all the lower dimensional skeleta can be contracted to a point, leaving only top dimensional spheres. $\square$

Remark: The perturbation trick for showing $\mathbb{T}_n$ is homotopy equivalent to a wedge of spheres for $n \leq 5$ does not obviously extend to all n. The problem is that a tiling may already contain a configuration that is a perturbation of a 4-valent vertex, so when we perturb a given 4-valent vertex, the boundary of the resulting cell will contain at least two terms where a 4-valent vertex is created. It's an interesting question of whether $\mathbb{T}_n$ is homotopy equivalent to a wedge of $(n-1)$-spheres in general.

REFERENCES

[1] **J.H. van Lint** and **R.M. Wilson**. *A course in combinatorics*. Cambridge University Press, Cambridge, 1992. xii+530 pp. ISBN: 0-521-41057-6; 0-521-42260-4