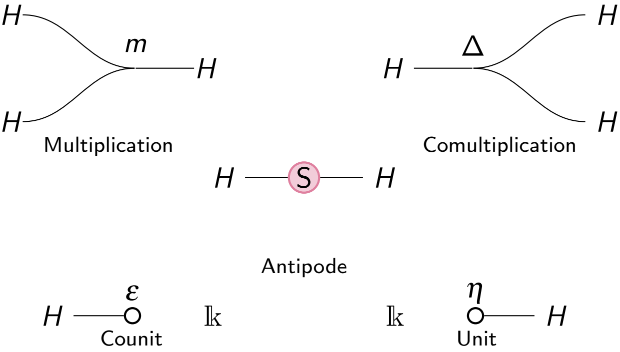
The Johnson Homomorphism and its Cokernel

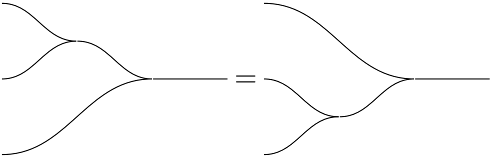
Jim Conant (partially joint with Martin Kassabov, Karen Vogtmann)

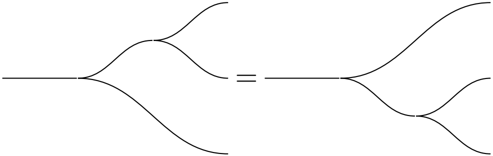
May 24, 2014

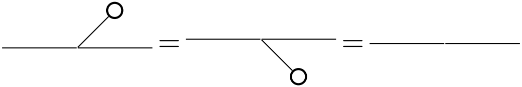
Appetizer: $\mathrm{Aut}(F_n)$ and cocommutative Hopf algebras

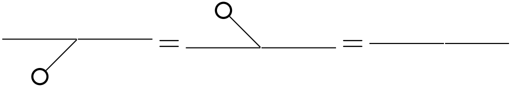
Let H be a cocommutative Hopf algebra.

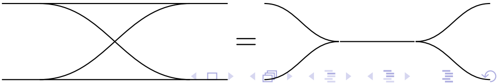


Associativity: 

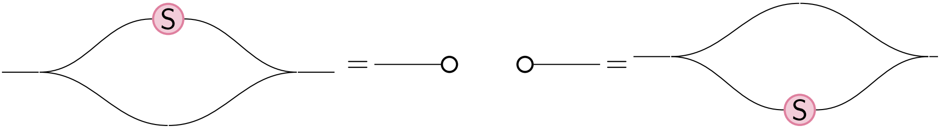
Coassociativity: 

$(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}:$ 

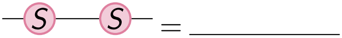
$m \circ (\eta \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes \eta) = \mathrm{id}:$ 

Compatibility of m and Δ: 

$$m \circ (\mathrm{id} \otimes S) \circ \Delta = \eta \circ \varepsilon = m \circ (S \otimes \mathrm{id}) \circ \Delta :$$



$S^2 = \mathrm{id}$:



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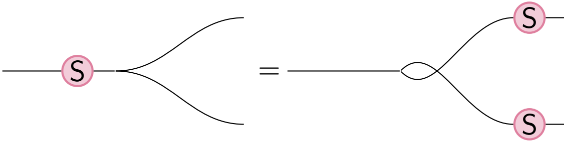
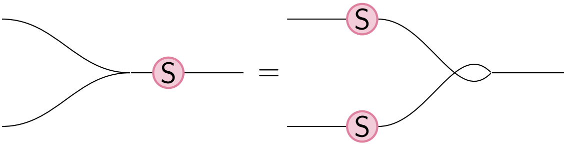
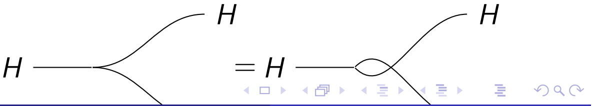


Compatibility of S and m:



Cocommutativity:



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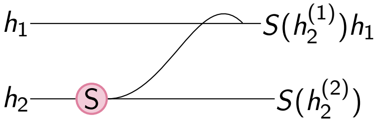
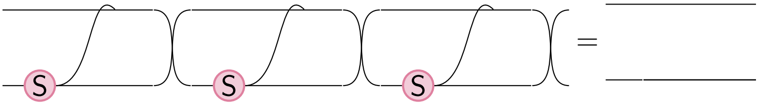
4. For example, if $\eta\colon F_2 \to F_2$ is defined by $x_1 \mapsto x_2^{-1}x_1$, $x_2 \mapsto x_2^{-1}$, then the action of η on $H^{\otimes 2}$ looks like:





Puzzle: Show $(\sigma_{12}\eta)^3 = \mathrm{id}$ using graphical calculus.

In order to show that $\mathrm{Aut}(F_n)$ acts in a well-defined way on $H^{\otimes n}$, one could take a presentation for $\mathrm{Aut}(F_n)$ and verify that all of the relations are satisfied via complex but fun graphical calculus arguments. There is also a more categorical way of doing it.

Definition

The Hopf algebra H acts on $H^{\otimes n}$ via *conjugation*. That is, suppose $h \in H$ and $\Delta^{2n}(h) = h_{(1)} \otimes h_{(2)} \otimes \cdots \otimes h_{(2n-1)} \otimes h_{(2n)}$, using Sweedler notation. Then define

$$h \star (h_1 \otimes \cdots \otimes h_n) = h_{(1)} h_1 S(h_{(2)}) \otimes \cdots \otimes h_{(2n-1)} h_n S(h_{(2n)}).$$

Let $\overline{H^{\otimes n}}$ be the quotient of $H^{\otimes n}$ by the subspace spanned by elements of the form

$$(h - \varepsilon(h) \cdot 1) \star (h_1 \otimes \cdots \otimes h_n),$$

i.e., this is the maximal quotient of $H^{\otimes n}$ where the conjugation action of H factors through the counit.

- Later on in the talk, the group

$$H^{2n-3}(\mathrm{Out}(F_n), \overline{T(V)^{\otimes n}})$$

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- Now you know how the action is defined, and the appropriate sense of suspense has been instilled!

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4. The Dehn-Nielsen map $\mathrm{Mod}(g,1) \to \mathrm{Aut}(\pi)$ induces a map $\mathrm{DN}_k\colon \mathrm{Mod}(g,1) \to \mathrm{Aut}(\pi/\pi(k))$. The Johnson filtration

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5. So $\mathbb{J}_1$ is the Torelli group.

Let $\mathsf{J}_k = \mathbb{J}_k/\mathbb{J}_{k+1} \otimes \mathbb{k}$. Then $\mathsf{J} = \bigoplus_{k\geq 1} \mathsf{J}_k$ is a Lie algebra and an SP-module.

Question

What is the Lie algebra and SP-module structure of J?

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4. Let $\varphi\in\mathbb{J}_1\subset \mathrm{Aut}(\pi)$, and let $\mathscr{B}$ be a standard symplectic basis for V. Then for every $b\in\mathscr{B}$, $\varphi(b)=b\alpha_b$ for some $\alpha_b\in\pi(2)$. So we can project α_b to lie in $\mathsf{L_2(V)}\cong\bigwedge^2\mathsf{V}$. Then

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5. V is symplectic, so there is a canonical isomorphism $V \cong V^*$. It turns out that $\mathrm{im}\,\tau$ is contained in the subset of $V \otimes \bigwedge^2 V$ spanned by elements $a \otimes (b \wedge c) + c \otimes (a \wedge b) + b \otimes (c \wedge a)$, which is a copy of $\bigwedge^3 V \subset V \otimes \bigwedge^2 V$. This gives rise to the *classical Johnson homomorphism*.

$$\tau_1 : \mathbb{J}_1 \twoheadrightarrow \bigwedge^3 V.$$

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Theorem (Hain 1997)

$\mathrm{im}(\tau)$ *is the Lie algebra generated by the image of elements in degree* 1. *I.e. by* $\bigwedge^3(V)$.

- Define $D_k(V)$ as the kernel of the bracketing map:

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- Another source of interest in the cokernel C_k is that Matsumoto and Nakamura showed there exist Galois obstructions in C_k related to the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In particular, Deligne's motivic conjecture implies that the degree k part of the free graded Lie algebra $\mathsf{L}(\sigma_3, \sigma_5, \sigma_7, \cdots)$ on odd generators embeds in C_{2k} (as a trivial SP-module).

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$$[2k, 2\ell]_{\mathsf{SP}} \otimes \mathscr{S}_{2k-2\ell+2} \subset \mathsf{C}_{2k+2\ell+2}$$
$$[2k+1, 2\ell+1]_{\mathsf{SP}} \otimes \mathscr{M}_{2k-2\ell+2} \subset \mathsf{C}_{2k+2\ell+4}$$
(C-Kassabov-Vogtmann 2013)

Low order calculations (Morita-Sakasai-Suzuki 2013)

$C_1 = C_2 = 0$

$C_3 = [3]_{SP}$

$C_4 = [21^2]_{SP} \oplus [2]_{SP}$

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Red classes are part of the families due to Morita, Matsumoto-Nakamura, Enomoto-Satoh, and CKV13.

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Theorem (C- 2013)

The coinvariants $V^{\langle k\rangle}_{D_{2k}}$ embed in C_k.

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Conjecture

The part of C_k with partitions of size k is isomorphic to $V_{D_{2k}}^{\langle k \rangle}$.

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Theorem (C-Kassabov 2014))

There is a map $\mathsf{C} \to H^{2n-3}(\mathrm{Out}(F_n); \overline{T(V)^{\otimes n}})$ *with "large" image. If* $H^{2n-3}(\mathrm{Out}(F_n); \overline{T(V)^{\otimes n}}) = \oplus_\lambda m_\lambda [\lambda]_{\mathrm{GL}}$*, then* $\operatorname{im}\mathrm{Tr}$ *contains* $\oplus_\lambda m_\lambda [\lambda]_{\mathrm{SP}}$*.*

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- $n = 2$: $H^1(\mathrm{GL}_2(\mathbb{Z}), \overline{T(V)^{\otimes 2}})$ contains the family

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and

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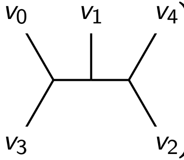
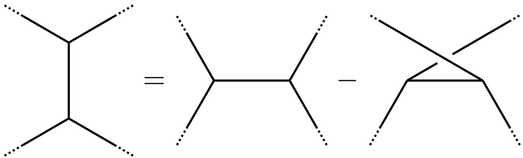
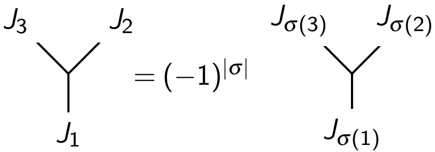
  yielding obstructions in C_{2k+3} and C_{2k+5}.
- Projecting $\overline{T(V)^{\otimes n}} \to S(V)^{\otimes n}$ recovers the CKV2013 obstructions.

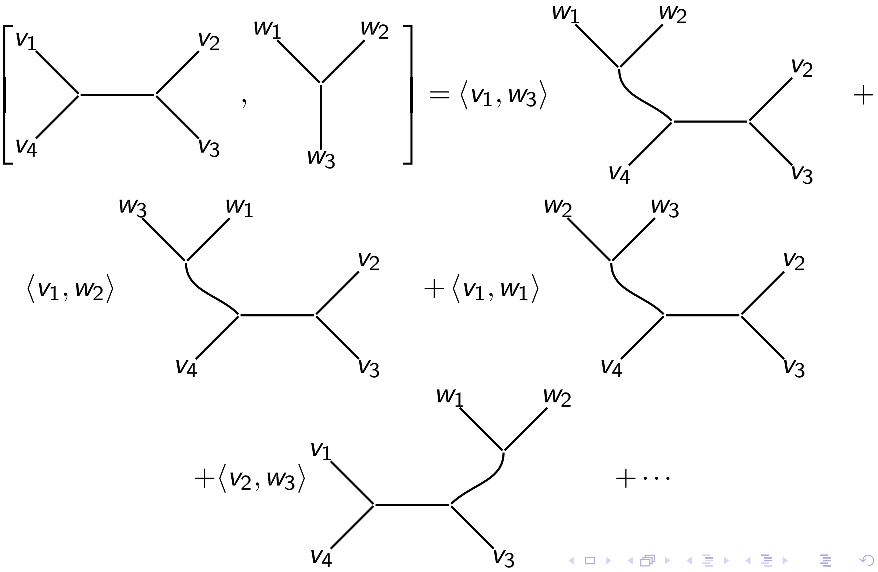
Tree interpretation of D(V)

$D_3(V) = \mathbb{Q}\Big\{$  $\Big\}$ $/\mathrm{IHX} + \mathrm{AS} + \mathrm{MultiLin}$

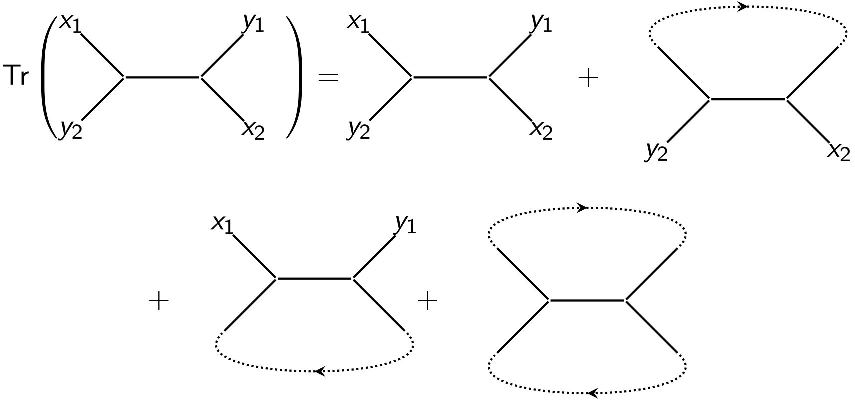
1. IHX: 

2. AS: 

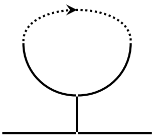
The bracket map

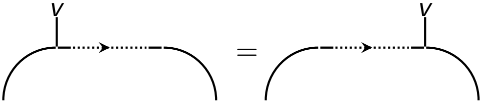


Our strategy is to graphically define a map Tr on D(V) and mod out by enough relations so that it vanishes on iterated commutators of degree 1 elements.

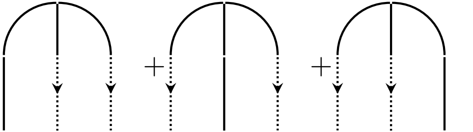


In general, the target of Tr is defined to be the space $C_1\mathscr{H}$, spanned by V-labeled trees with some univalent vertices connected by directed edges, modulo IHX, AS and Multilinearity of the trees, and switching edge order giving a sign. We quotient $C_1\mathscr{H}$ by the following relations:

1. 
   $= 0$ (Lollipop)

2. 

   (Slide)

3. 
   $= 0$ (Jellyfish)

Let $\Omega(V) = C_1\mathscr{H}/\text{Lollipop} + \text{Slide} + \text{Jellyfish}$.

Theorem

$\mathrm{Tr}\colon \mathrm{D}(V) \to \Omega(V)$ *vanishes on* $\mathrm{im}(\tau)$*, so induces an invariant of the cokernel.*

Proof.

1. Let t be a degree 1 tree.

□

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1. Let t be a degree 1 tree.
2. Show $\mathrm{Tr}[t, X] = [t, \mathrm{Tr}(X)]$.

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2. Show $\mathrm{Tr}[t, X] = [t, \mathrm{Tr}(X)]$.
3. Done by induction.

□

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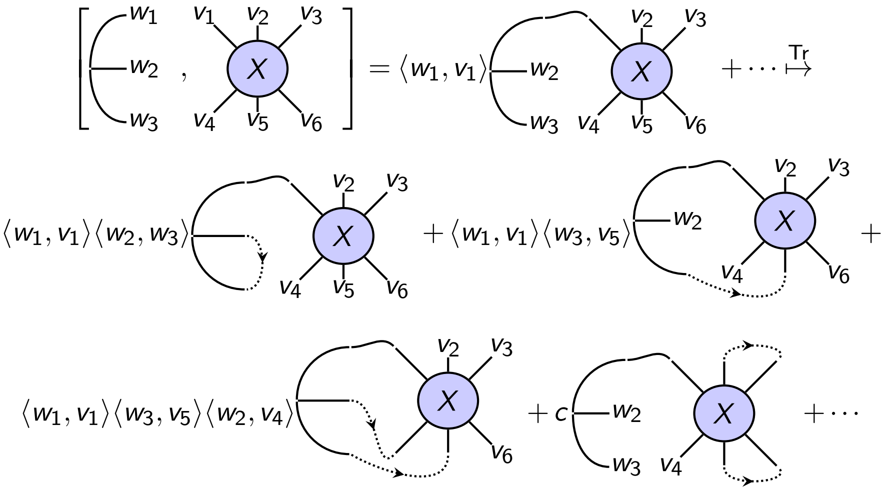
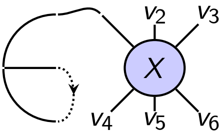
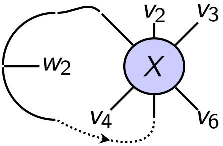
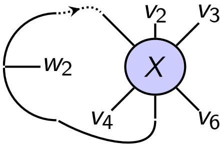


The terms of the form 

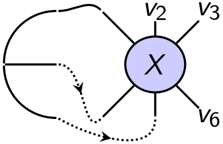
are zero by the Lollipop relation.

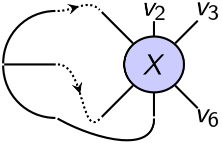
Terms of the form 

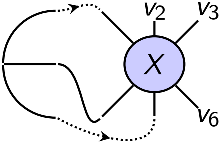
cancel with terms of the form



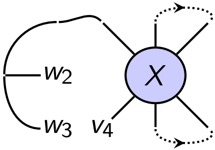
via the Slide relations. Terms of the form



cancel with 

and 

via the Jellyfish relation.

The leftover terms, like  are part of $[t, \mathrm{Tr}(X)]$. Thus we have shown that $\mathrm{Tr}([t, X]) = [t, \mathrm{Tr}(X)]$.

Now break up $\Omega(V)$ into graded pieces:

$$\Omega(V) = \bigoplus_{r,s} \Omega_{r,s}(V),$$

where r is the rank of the graph (or the number of external edges) and s is the number of V-labeled hairs.

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3. Theorem: Tr is onto $\Omega_{r,s}\langle V\rangle$, which follows from C-Kassabov-Vogtmann 2013.
4. If $r \geq 2$ then $\oplus_{s\geq 0}\Omega_{r,s}(V) \twoheadrightarrow H^{2r-3}(\mathrm{Out}(F_r), \overline{T(V)^{\otimes r}})$. (C-Kassabov 2014)

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3. Are the Galois obstructions related to the Morita classes in $\mu_k \in H^{4n}(\mathrm{Out}(F_{2n+2}); \mathbb{Q})$? Both first appear in degree 6.