Vassiliev Invariants and Embedded Gropes

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Abstract

It has often been wondered what type of geometric data finite type invariants can detect. The answer is not obvious, their definition being combinatorial in spirit. In this paper we show that these invariants can be interpreted as obstructions to a certain natural geometric question: does the knot bound an embedded tower of surfaces into the three sphere? This is a question which arises naturally in the theory of hierarchies of 3-manifolds. The towers of surfaces we consider are called gropes, and can be thought of as a topological manifestation of the lower central series.

Gropes can also be thought of as representing a particular type of clasper surgery in the sense of Habiro. In the last section of the paper we derive some surprising results about claspers which are hopefully of independent interest.

This paper is a revised version of a previously posted paper. The last section relating to claspers is completely new.

Contents

1 Introduction

The precise statement of our result is as follows:

Theorem 1.1 *If a knot bounds an embedded grope of class n in S^3, then Vassiliev invariants up to degree $\lceil \frac{n}{2} \rceil$ taking values in any abelian group do not distinguish that knot from the unknot.*

together with the optimality result:

Theorem 1.2 *For all n, there are knots bounding embedded gropes of class n in S^3 which are distinguished from the unknot by a type $\lceil \frac{n}{2} \rceil + 1$ invariant.*

One might ask why $\lceil \frac{n}{2} \rceil$? As a grope of class n represents an n-commutator, the work of Ng and Stanford[NS] on n-triviality of group elements would suggest that $n-1$ is the natural answer. The problem is that the natural moves that implement "letter deletion" as considered in that paper cannot be realized independently. The factor of 2 that results is the same factor of 2 encountered by Goussarov [Go2], where he considers non-independent moves. In fact his work directly implies that knots bounding embedded class n gropes have trivial invariants up to degree $\lfloor \frac{n}{2} \rfloor - 1$, by using, say moves of type II (see section 3.2) without the restriction on the vertices.

It is not hard to show that the Alexander polynomial of a knot bounding a grope of class ≥ 3 is trivial. Since the Alexander polynomial has associated to it finite type invariants of arbitrarily large type, a converse to theorem 1.1 is not possible. In [CT] the notion of *grope cobordism* is introduced. The analogous condition to bounding a grope in this language is that the knot cobounds an annulus-like grope with the unknot. The unknot may link with the knot geometrically. It turns out that the analog of theorem 1.1 holds with $\lceil \frac{n}{2} \rceil$ replaced by $\lfloor \frac{n}{2} \rfloor$, and that a sort of converse holds: that if two knots share type n invariants, they cobound an embedded grope of class $n+1$.

An interesting consequence of the main theorem is that a knot bounding a grope of arbitrarily large class cannot be distinguished from the unknot by finite type invariants. It is a conjecture of Mike Freedman that this phenomenon is impossible. More precisely he conjectures the related statement that in any three manifold, you cannot have an infinite embedded grope, every stage of which is incompressible. In the case that the three manifold is a knot complement, theorem 1.1 therefore reduces Freedman's conjecture to the famous open question of whether finite type invariants detect knottedness.

The definition of a finite type invariant is that a certain alternating sum over subsets of crossing changes vanishes. This definition can be generalized by replacing the phrase "crossing changes" by the word "homotopies." See section 1.1 for a precise statement.

The way we prove theorem 1.1 is to construct certain homotopies, or *moves*, on the embedded grope. These moves restrict to moves of the boundary and so can be plugged into the finite type definition. The resulting relations allow us to keep writing the values of the invariants in terms of their values on simpler and simpler knots. The moves we use are relatively obvious except in the case of the in-out trick. The in-out trick is a certain pair of moves, and has some interesting consequences for the theory of claspers. (See section 7). The majority of the paper is devoted to setting up these moves and analyzing their effect. In section 6, theorem 1.1 is proved in a couple of pages using the moves and the resulting alternating sum relations on the finite type invariants. In section 7, we prove theorem 1.2, using the moves, Habiro's clasper theory and the calculation of the Jones weight system derived by Bar-Natan.

This line of research was stimulated by conversations with Mike Freedman and Peter Teichner, and also by the work of Lin and Kalfagianni [LK]. I also wish to thank the referee for much helpful advice.

1.1 A slight reformulation of the finite type axiom

For a cheerful introduction to the theory of finite type invariants see Bar-Natan's original paper [B-N].[1]

We would like to make a slight reformulation of the definition of a type n invariant which will make our arguments easier to state later. The new (but equivalent) definition is also aesthetically pleasing in that it makes sense for maps of any topological space into any other.

Suppose $h : S^1 \times I \to S^3$ is a homotopy of a knot to another (embedded) knot. Let $\mathrm{Supp}(h) \subset S^3$, the *support* of h, be the image of $M \times I$ where $M \subset S^1$ is the closure of all points which h does not fix. Two homotopies of the same knot are said to be disjoint if their supports are disjoint.

Definition 1.1 *Let ν be an abelian group valued knot invariant. ν is said to be of* type n *if for every knot $K \subset S^3$ and every collection of $n+1$ disjoint homotopies $\{h_i\}$ of K, we have the following:*

$$\sum_{\sigma \subset \{1,\dots,n+1\}} (-1)^{|\sigma|} \nu(K_\sigma) = 0$$

Here $|\sigma|$ is the cardinality of σ and K_σ is the knot modified by the homotopies $\{h_i | i \in \sigma\}$.

We will call a set of disjoint homotopies of a knot a *scheme*, which abbreviates Goussarov's term "variation scheme."

If one fixes a planar diagram of the knot, a crossing change is a particular type of homotopy. Such a homotopy pushes the top arc at a crossing down through the bottom arc. Because of this, a type n invariant in the sense of definition 1.1 is a type n invariant in the usual sense.

[1] Many of the conjectures contained in Bar-Natan's paper have now been resolved in the negative.

Conversely, one observes that an arbitrary homotopy of S^1 in any 3-manifold which begins and ends with an embedding is itself homotopic to a collection of disjoint finger moves.[2] (One just puts the homotopy in general position.) But any finger move is a crossing change in some planar diagram of the knot. (Contract the arc which guides the finger move until it is very short and vertical in the planar projection.) Hence in definition 1.1 one can substitute the phrase "collections of crossing changes" for the word "homotopy". That type n invariants vanish on the alternating sums arising from $n+1$ collections of crossing changes is well known. See [Go] lemma 5.2 for a proof.

A piece of terminology: A knot is said to be *n-trivial* if all type n invariants taking values in any abelian group do not distinguish it from the unknot.

Convention: All our Vassiliev invariants will vanish on the unknot. This does not affect our generality because every type m invariant is a constant (type 0 invariant) plus a type m invariant vanishing on the unknot.

1.2 Gropes

Gropes are certain 2-complexes formed by gluing punctured surfaces together. (A punctured surface is a surface with an open disk deleted.) They can be defined recursively using a quantity called *depth*. There is an anomalous case when the depth is 1: the unique grope of depth 1 is a circle. A grope of depth 2 is a punctured surface. To form a grope, G, of depth n, first prescribe a *symplectic basis* $\{\alpha_i, \beta_i\}$ of a punctured surface, F. That is, α_i and β_i are embedded curves in F which represent a basis of $H_1(F;\mathbb{Z})$ such that the only intersections among the α_i and β_i occur when α_i and β_i meet at a point. (α_i and β_i represent dual classes.) Glue gropes of depth $< n$ to each α_i and β_i with at least one such added grope being of depth $n-1$. (Note that we are allowing any added grope to be of depth 1, in which case we are not really adding a grope.)

Definition 1.2 *The surface $F \subset G$ is called the* bottom stage *of the grope.*

Definition 1.3 *The* tips *of the grope are those prescribed symplectic basis elements of the various punctured surfaces of the grope which do not have gropes of depth > 1 attached to them.*

For instance in figure 1 there are 9 tips.

Depth is just a tool for defining gropes. The *class* of the grope is more important theoretically. It is defined recursively as follows.

Definition 1.4 *The class of a depth 1 grope is 1 and that of a depth 2 grope is 2. Suppose a grope G is formed by attaching the gropes of lower depth $\{A_i, B_i\}$ to a symplectic*

[2] A finger move is a homotopy of the knot guided by a framed arc from the knot to itself: one pushes a little finger of the knot along one end of the arc until it crashes through the part of the knot at the other end of the arc.

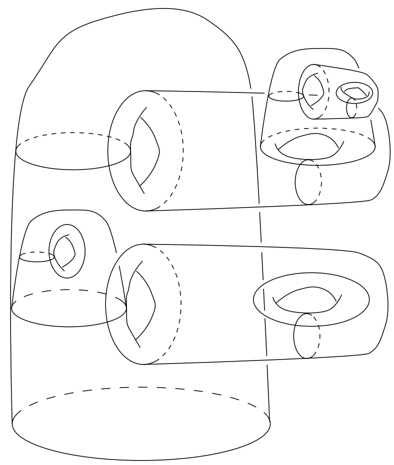


Figure 1: A grope of class 4 and depth 5.

basis $\{\alpha_i, \beta_i\}$ *of the bottom stage, such that* $\partial A_i = \alpha_i$,$\partial B_i = \beta_i$. *Then* $class(G) = \min_i\{class(A_i) + class(B_i)\}$.

For a class k grope G, the boundary ∂G represents an element of $\Gamma^k\pi_1(G)$, the kth term of the lower central series of the free group $\pi_1(G)$. For instance, in figure 1, ∂G is a commutator of the form $[x,[y,[z,[t,u]]]] \cdot [[a,b],[c,d]]$, where each letter represents a tip. (It can, in fact, also be shown that ∂G does not represent an element lying in $\Gamma^{k+1}\pi_1(G)$[FT].) A fundamental aspect of gropes is the following, which helps explain their interest: If X is a topological space, $\sigma \in \pi_1(X)$ lies in $\Gamma^k\pi_1(X)$ if and only if any representative of σ extends to a map of a grope of class k into X [FT]. Restricting to embedded gropes therefore gives a natural geometric strengthening of the concept of a k-commutator., Embedded gropes have been especially useful in 4-manifold topology. They appear throughout Freedman and Quinn's book [FQ] and also feature prominently in recent work of Cochran, Orr and Teichner[COT] concerning a filtration of the knot concordance group. They also appear in [FT], [KT], and [K]. One of the interesting aspects of the present paper is that it establishes a connection of the theory of embedded gropes to 3 dimensional topology.

Finally, we'd like to give a couple of examples of embedded gropes.

First, the kth iterated untwisted Whitehead double, $Wh^k(K)$, of any knot K bounds an embedded grope of class $k+2$. (As usual, the notation $Wh^k(\cdot)$ is ambiguous as there are two possible untwisted Whitehead doubles of any knot. However, the statement holds no matter what set of k choices one makes.) Figure 2 attempts to illustrate this for the case $k = 1$. Here S is a Seifert surface for the original knot K, and is genus 1 in the picture. The left-hand side of figure 2 is the embedded version of the abstract model on the right. The left-hand side of figure 2 depicts the action near the "clasp part" of the Whitehead double.

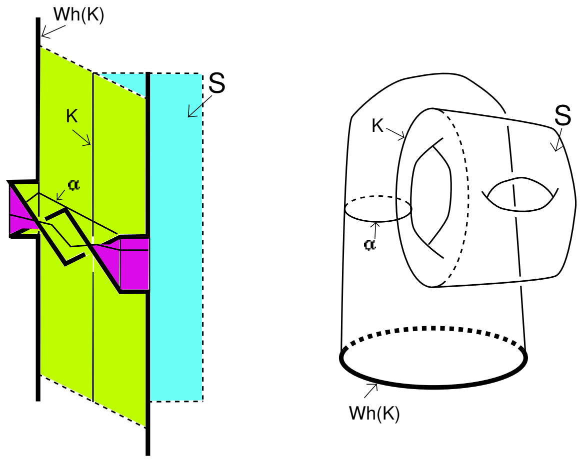


Figure 2: The Whitehead double of a knot bounds a grope of class 3.

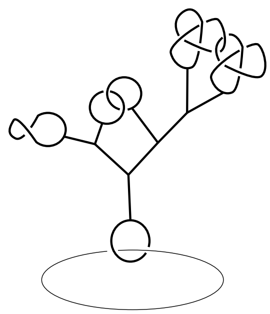


Figure 3: An embedded grope comes from a particular type of clasper surgery.

The iterated Whitehead double is not particularly representative of the general case because most of the tips of such a grope do not link geometrically, as would be the case in general. (To anticipate what comes later, the associated graph will have at most three edges, depending on the Seifert surface of the original knot K.) In fact, it is well-known that the Whitehead double of an n-trivial knot is $n+1$-trivial. It follows that $Wh^k(K)$ is $k+1$-trivial, about twice what our main theorem would yield.

Secondly, it is relatively easy to prove [CT] (also see section 7) that clasper surgeries [H1,H2] of a very special form give rise to embedded gropes of class k. Namely a tree clasper surgery on the unknot, where the clasper has $k+1$ leaves all of which avoid the spanning disk of the unknot except for one leaf called the root leaf which forms a meridian to the unknot. See figure 3. Indeed the non-root leaves correspond to the tips of the grope. This is in fact the most general embedded grope where each surface stage is of genus one. For higher genus, one would add "boxes" to the clasper. A general tree clasper surgery gives rise to a *grope cobordism*, a concept which is explored in [CT].

1.3 The work of X.S. Lin and E. Kalfagianni

The main theorem of this paper is similar to and inspired by that of a preprint of X.S. Lin and E. Kalfagianni[LK]. The main theorem of that paper is that knots which bound certain immersed gropes of height $n+2$ are $l(n)$-trivial, where $\lim_{n\to\infty} l(n) = \infty$. More specifically, they consider immersed gropes such that all self-intersections occur away from the bottom stage. There is also the restriction that the bottom stage is *regular*, which among other things implies that the complement of the Seifert surface has free fundamental group. (It should be noted that the obvious generalization of their and my result, that all knots bounding immersed gropes are to some degree trivial, meets with the problem that all knots bound immersed gropes of arbitrary class, since the lower central series of a knot complement stabilizes after the second term.) Their method of proof is to find crossing changes which implement the group-theoretic $n-1$-triviality (as defined in [NS]) of an n-commutator.

It turns out in their case that one can not fully realize the degree of triviality present in an n-commutator, the problem being the same as with the present case in that one must be able to find n geometric independent moves which have the effect of deleting a letter in the commutator. Examples due to the author suggest that at most a logarithmic function of n of the moves can be realized, and indeed the function $l(n)$ which they find is logarithmic.

2 Grope foundations

Suppose a tip of an embedded grope bounds an embedded disk into the grope complement. Then one can delete an annular neighborhood of the tip and glue in two parallel copies of the disk. This procedure is usually abbreviated by the term "surgery on the disk." This has the effect of reducing the genus of the surface stage to which the tip belongs by one. Hence if the stage were already a punctured torus, it would become a disk under this operation. In this case we can iterate the procedure, reducing the genus of the next stage down. If each stage of an embedded grope is of genus one, and if a tip bounds a disk into the embedded grope complement, then this procedure constructs a spanning disk for the boundary knot.

In the case when the stages of a class n grope are possibly of higher genus, it is straightforward to show that there is nevertheless a partition of the tips into n sets of tips, such that if one of these sets of tips bounds disks into the embedded grope complement, then iterated surgery on these disks provides a spanning disk for the knot.

Definition 2.1 *A set of tips of an abstract grope, $\mathcal{G}$, has the* trivialization property *if, for each emebedding $e : \mathcal{G} \to S^3$, where these tips bound disks into $S^3 \backslash e(\mathcal{G})$, the above iterated surgery construction produces a spanning disk for $e(\partial \mathcal{G})$. Equivalently, if one deletes the letters in $\pi_1(\mathcal{G})$ corresponding to these tips from the word $\partial \mathcal{G}$, then this word*

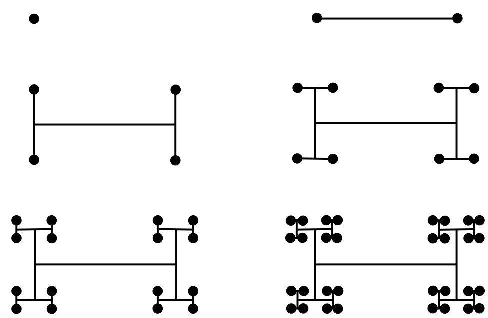


Figure 4: The 1-complexes Ξ_0 to Ξ_5

trivializes.

For instance in figure 1, we observed that the boundary is the curve $[x, [y, [z, [t, u]]]] \cdot [[a, b], [c, d]]$ where each letter represents a tip. In this case, $\{\{x, a\}, \{y, b\}, \{z, c\}, \{t, u, d\}\}$ is a partition into four sets of tips with the trivialization property.

We now proceed to find a nice handlebody surrounding the embedded grope. It is not hard to see that a grope deformation retracts onto its 1-spine, so that a regular neighborhood of an embedded grope in S^3 is indeed a handlebody. However it will be convenient to keep more careful track of how the grope sits inside the handlebody. For instance, theorem 2.1 will allow us to assume that the slice of the grope that sits in a cross-section of a handle looks like one of the following family of 1-complexes, $\{\Xi_i\}$.

Definition 2.2 *Ξ_i is defined recursively as follows. Ξ_0 is a point. Suppose that Ξ_i has been defined and has 2^i univalent vertices. Form Ξ_{i+1} from Ξ_i and 2^i intervals by gluing each univalent vertex of Ξ_i to the midpoint of an interval. See figure 4.*

Before stating the theorem, we'd like to explain less formally what's going on, by considering what happens for a grope of class 3 both of whose surface stages are genus one. Think of each of these surfaces as a disk with a pair of dual bands attached. To form the grope one glues the core of one of the bands of one of the surfaces to the boundary of the other surface. The result is pictured in figure 5. One can think of the right-hand side of this picture as a standard unknotted embedding of the grope into S^3. The regular neighborhood of this embedding is a genus 3 handlebody. The cross-section of one handle is just Ξ_1 and the cross-section of the other two handles is Ξ_2. In this picture there are three obvious tips forming the cores of the three handles. Observe that there are annuli which extend from these tips disjointly to the handlebody's exterior. For instance in the left-most Ξ_2 handle, one extends the annulus upward from the page, whereas in the other Ξ_2 handle one extends the annulus downward from the page.

In the general case, it is convenient to have such annuli, (which we call *pushing annuli*), which disjointly extend from each tip to the surface of the handlebody and allow us to

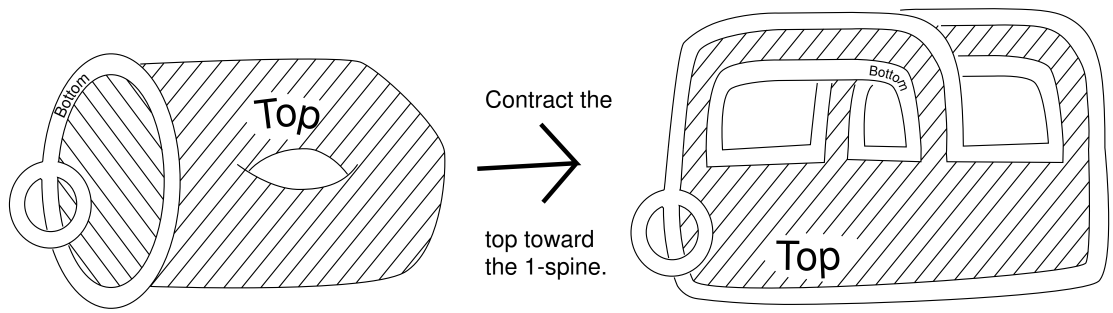


Figure 5: A grope of class 3.

push out a tip to a curve on the surface of the handlebody. Drawing pictures like 5 will convince the reader of the truth of the following theorem, although it is proved in detail for the sake of completeness.

Theorem 2.1 *Let $G \subset S^3$ be an embedded grope. Then there is a handlebody which is a ball Σ with handles $\{H_i = D^2 \times I\}$ with the following properties:*

(i) $\Sigma \cap H_i = D^2 \times \partial I$

(ii) $\Sigma \cup \bigcup_i H_i$ *is a regular neighborhood of* G.

(iii) $\forall t \in I$ *the cross-section* $G \cap (D^2 \times \{t\}) \subset H_i$ *is* Ξ_l *for some* l *depending on* i.

(iv) $v_i \cap (D^2 \times \{t\}) \subset H_i$ *is the midpoint of* $\Xi_1 \subset \Xi_l$.

(v) There exist embedded pushing annuli P_i *inside the handlebody where* $\partial P_i = v_i \amalg \overline{v_i}$ *for some curve* $\overline{v_i}$ *on the surface of the handlebody. The pushing annuli* P_i *also satisfy:*

(a) $P_i \cap G = v_i$

(b) $int(P_i) \cap int(P_j) = \emptyset$ *for* $i \neq j$.

[Proof]

We construct the handlebody for an unknotted model of G, which we also call G, in S^3. Since it will be a regular neighborhood, the handlebody with all its decorations will also be present for any embedding of G in S^3. We proceed by induction.

In the case G is of depth 1, let the model of G be an unknot. Σ is a small ball around a point of G, and the single handle is a regular neighborhood of the portion of the arc of G that lies outside Σ. The pushing annulus P_1 is just the intersection of a spanning disk for G with the handlebody.

For the inductive step, suppose G is a grope formed by gluing gropes $\{A_i, B_i\}_{i=1}^g$ to a symplectic basis $\{\alpha_i, \beta_i\}_{i=1}^g$ of the bottom stage F. Each A_i and B_i is of lower depth than G, so that by induction, for some models of A_i, B_i embedded disjointly in S^3, there are balls $\{\Sigma_{A_i}, \Sigma_{B_i}\}_{i=1}^g$ together with handles $\{H_{ij}^A\}_{j=1}^{m(i)}$ attached to each Σ_{A_i} and handles

$\{H_{ij}^B\}_{j=1}^{n(i)}$ attached to each Σ_{B_i}, which satisfy the conclusions of the theorem. Denote the pushing annuli P_{ij}^A and P_{ij}^B associated to the handles H_{ij}^A and H_{ij}^B respectively. Let $\partial P_{ij}^A = v_{ij}^A \amalg \overline{v_{ij}^A}$ and similarly for P_{ij}^B. (Our convention is that the curve with a bar is the pushed-out version of the unbarred curve.)

By modifying these we will construct a model of G in S^3 with an appropriate handlebody. First, modify the balls $\{\Sigma_{A_i}, \Sigma_{B_i}\}$ so that a piece of the bottom stage of A_i or B_i "sticks out" into S^3 in the following way. Choose a subarc δ_{A_i} of ∂A_i inside Σ_{A_i}. (Recall $\partial A_i = A_i$ if $\text{depth}(A_i) = 1$.) If $\text{depth}(A_i) > 1$, let ϵ_{A_i} be an arc which is a parallel push-off rel. endpoints of δ_{A_i} into the bottom stage of A_i. If $\text{depth}(A_i) = 1$, let ϵ_{A_i} be a parallel push-off rel. endpoints into the pushing annulus P_{i1}^A. In either case, let S_i be a sphere inside Σ_{A_i} which intersects $A_i \cup \bigcup_j P_{ij}$ exactly as the arc ϵ_{A_i}. Delete the inside of S_i from Σ_{A_i} and also delete an open neighborhood of an arc which connects S_i, inside of Σ_{A_i}, with the surface of the handlebody $\Sigma_{A_i} \cup \bigcup_j H_{ij}^A$ away from $A_i \cup \bigcup_j P_{ij}^A$. See figure 6. We can also do this procedure for B_i.

Via this procedure, we obtain new handlebodies which surround the embedded gropes $\{A_i, B_i\}$ except for small subdisks of the bottom stage which jut out. (In the depth 1 case it is a subdisk of the pushing annulus which sticks out.)

The first step in creating the bottom stage of G is to glue annuli along their cores to each ∂A_i and ∂B_i, perpendicularly to the bottom stages, or to the pushing annuli in the depth 1 case. These annuli can be arranged to only jut out of each $\Sigma_{A_{i_0}}$ and $\Sigma_{B_{i_0}}$ along the arc $\delta_{A_{i_0}}$ or $\delta_{B_{i_0}}$, and can be arranged to be disjoint, away from the attaching curves, from $\{A_i, B_i, P_{ij}^A, P_{ij}^B\}$. If a handle $H_{i_0 j_0}^A$ or $H_{i_0 j_0}^B$ had a cross-section of Ξ_N then after attaching a perpendicular annulus, the new cross-section is Ξ_{N+1}.

The next step is to plumb each A_i annulus together with the B_i annulus outside of Σ_i^A and Σ_i^B to form a punctured torus F_i for each $i = 1 \dots g$. The core of the A_i annulus will be α_i and that of the B_i annulus will be β_i. As pictured in figure 7 this may be done so that the pieces of surfaces which jut out (whether they be bottom stages or pushing annuli) remain with disjoint interiors.

Finally we form the bottom stage F by connecting the punctured tori $\{F_i\}$ with bands disjoint from everything except at their ends: run a band from F_1 to F_2, from F_2 to F_3 $\dots$, from F_{g-1} to F_g. The result is the embedded grope G, some of which lies in some handlebodies. To complete the induction we will take these handlebodies and form a new one completely surrounding G. The ball Σ is taken to be the union of the $\Sigma_{A_i}, \Sigma_{B_i}$ together with a regular neighborhood of the things that stick out of the balls: the plumbed sections of annuli, the bands, and the jutting-out bottom stages or pushing annuli of $\{A_i, B_i\}$. This added neighborhood may be assumed disjoint from the pushed-out tips $\{\overline{v_{ij}^A}, \overline{v_{ij}^B}\}$. The handles are the collection of all the handles $\{H_{ij}^A, H_{ij}^B\}$. Again see figure 7. $\square$

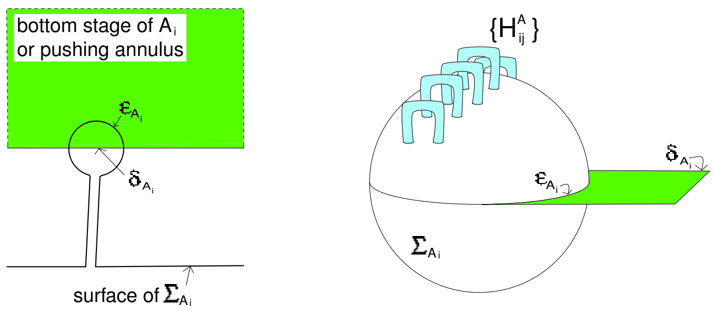


Figure 6: Pushing a piece of the grope or pushing annulus out of the ball.

2.1 The graph $\Gamma(G)$

Our task now is to construct a graph associated to an embedded grope which keeps track of the interactions between tips, or, more precisely, the handles of which they form the core.

First off, given an embedded grope, G, of class n, fix a partition of the set of tips into sets with the trivialization property (definition 2.1) denoted $V_1 \dots V_n$. As a matter of notation we define $\overline{V_i}$ to be the collection of pushed-out tips of V_i.

In the case when the embedded grope has bottom stage of genus 1, we observe the following convention. Notice that there are two "halves" of the grope: the embedded subgropes attached to the bottom stage. Each collection of tips V_i can be chosen to lie on one half of the embedded grope or the other. Choose the ordering so that every V_i on one half and every V_j on the other half satisfy $i < j$.

A collection V_i is said to be *framed unlinked* if each $\overline{v} \in \overline{V}_i$ bounds a disk whose interior intersects the grope only at handles not associated to a tip in V_i. This set of disks is called a *cap*. (When a disk does intersect a handle, by general position we can assume it does so in a single level $D^2 \times \{t\}$ for each connected component of intersection.) If V_i is not framed unlinked, we say it is *framed linked*. The reason for this terminology is that even if a collection of handles $\{H_i\}$ looks like an unlink, some pushing annulus may have nontrivial framing and so a pushed-out tip $\overline{v}$ may link with v and hence will not be able to bound a disk into the grope complement.

Secondly, fix a generic projection (a homeomorphism $S^3 \backslash \{\infty\} \cong \mathbb{R}^2 \times \mathbb{R}$) so that the 1-manifolds with boundary, $\overline{V}_i \cap \Sigma$, together with the attaching regions for the handles, are standardly arranged in decreasing order as the height function associated to the projection increases as in figure 8. This is a side view of the ball, whereas usually in projections the view is from above. It is reasonably clear that by sliding around the handles over the ball we can put the embedded grope with its handlebody in the given position with respect to the height function.

With these additional data fixed we can now define our graph.

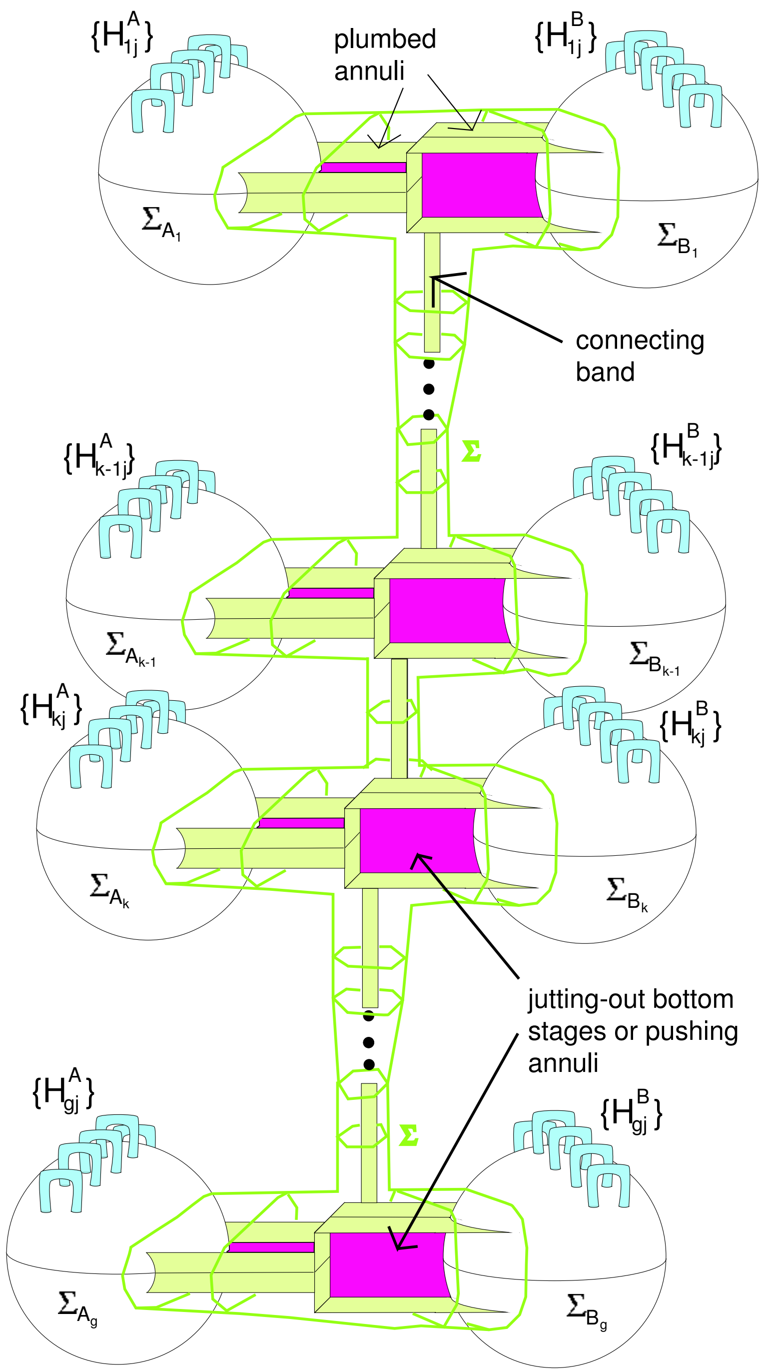


Figure 7: Forming the handlebody surrounding the grope.

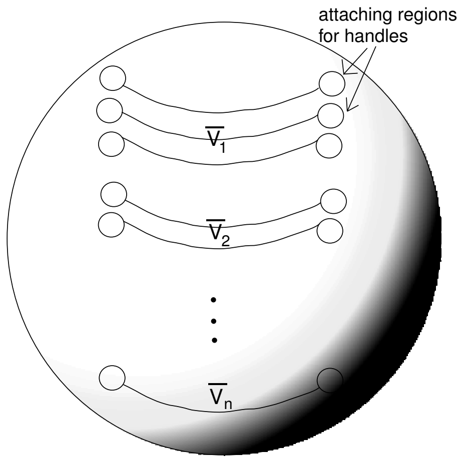


Figure 8: Standardly arranged attaching regions.

Definition 2.3 *Let $\Gamma(G)$ be the graph which has vertices $V_1, \dots V_n$, and whose edges are given as follows:*

If V_i is framed linked, put an edge from V_i to itself.

If a pushed-out tip $\overline{v}_i \in \overline{V}_i$ crosses over a pushed-out tip $\overline{v}_j \in \overline{V}_j$ with $i > j$, put an edge between V_i and V_j. Here "crosses over" refers to the fixed projection.

An example is shown in figure 9. In the picture, we are looking down on the handlebody with respect to the height function. $\overline{v}_3$ crosses over $\overline{v}_1$ so there is an edge. $\overline{v}_1$ is knotted, so there is a loop. $\overline{v}_3$ is unknotted but has a nontrivial framing, hence there is a loop at this vertex also.

Back in the general setting, as an exercise note that if V_1 is an isolated vertex, then the knot ∂G is trivial. In this case, possibly after an isotopy, there is a level plane $\mathbb{R}^2 \times \{t_0\}$ which separates the handles associated to the tips in V_1 from all the other handles. By hypothesis the pushed-out tips in $\overline{V}_1$ bound disks which only hit the handles below the level plane. But these disks can be surgered to lie above the plane using an innermost disk argument. Beware that in general an isolated vertex does not imply triviality because it does not imply that the handles associated to that vertex can be squeezed between level planes away from the other handles. This is the case, for instance, in figure 9. $\overline{v}_2$ does not bound a disk into the handlebody complement.

One final definition before closing the section.

Definition 2.4 *A collection of vertices $V_{i_1}, \dots, V_{i_k}$ is said to be* free *if for all $1 \le s, t \le k$ there is no edge in the graph connecting V_{i_s} with V_{i_t}. (In particular, framed linked vertices are excluded.)*

An example is shown in figure 10.

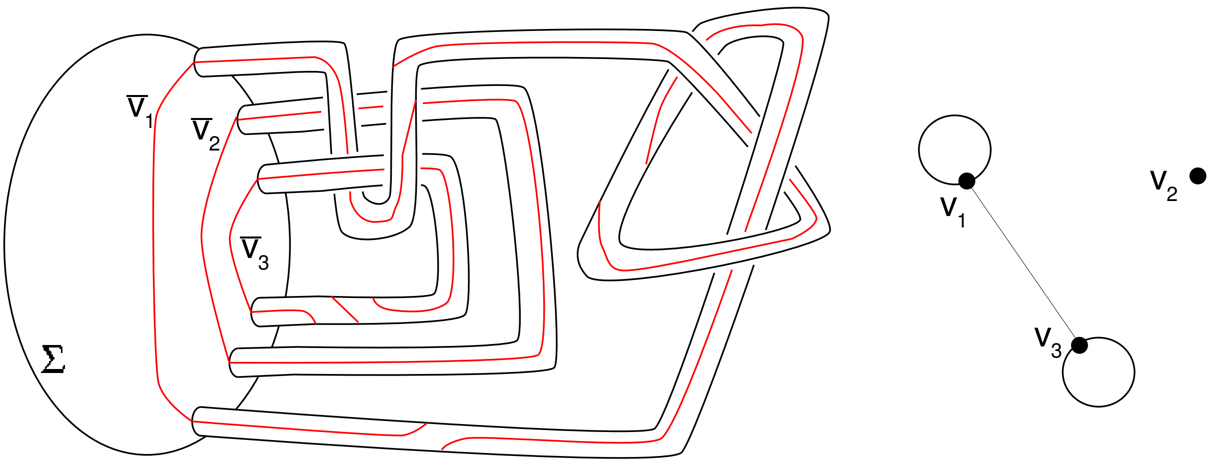


Figure 9: The regular neighborhood of the grope and its associated graph.

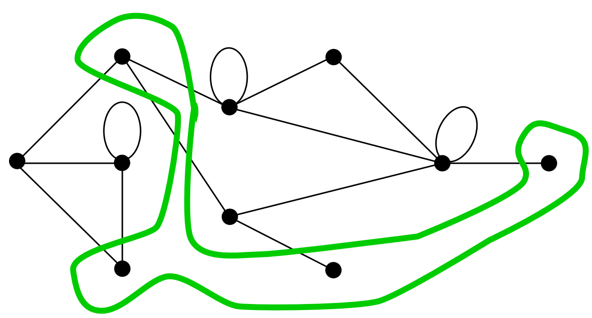


Figure 10: The circled vertices are a free collection of 4 vertices.

3 The easy moves

Recall that the term "move" is synonomous with "homotopy." The moves in this section will take an embedded grope to another embedded grope. When restricted to the boundary they give a homotopy of a knot to another knot.

3.1 Move type I, Killing an edge

Given an edge in the graph Γ, we define a move which has the effect of deleting the edge. Suppose the edge is between V_i and V_j where $i < j$. That means that some of the handles in V_j cross over some of the handles in V_i. Then the move is defined to be the homotopy which switches these handle crossings, supported in balls which are regular neighborhoods of the guiding arc of the crossing change. Next we describe how to remove an edge from V_i to itself. To unknot a handle in V_i, first do handle crossings of the handle with itself so that the handle bounds a disk which intersects only other handles. However we must also make sure the handle is untwisted, which is to say that the pushed-out tip of the handle bounds a disk which intersects only other handles. So Dehn twist to remove the appropriate number of multiples of the meridian of the handle. This twist is a homotopy supported in some small section of the handle $D^2 \times [a, b]$. Do this for every handle in V_i to remove the edge.

Notice that any number of type I moves may be performed simultaneously, since the supports are by construction disjoint, with the effect that the corresponding edges are deleted in Γ.

3.2 Move type II, Moves on free sets of vertices

Given a free set of k vertices, $\mathcal{F}$, we define k moves as follows. Since the set of vertices is free, there are level planes which separate the collections of handles associated to the tips in $\mathcal{F}$, and which intersect the ball Σ in circles which lie standardly as level circles between the attaching regions of the collections of handles. We can now choose homotopies supported between the appropriate planes which contract the sets of handles toward the ball to little handles whose pushed-out tips bound embedded disks. See 11. These moves obviously have disjoint support by construction, and further doing any collection of them has the effect of trivializing at least one set of handles with the trivialization property. This has the effect of unknotting the boundary of the embedded grope.

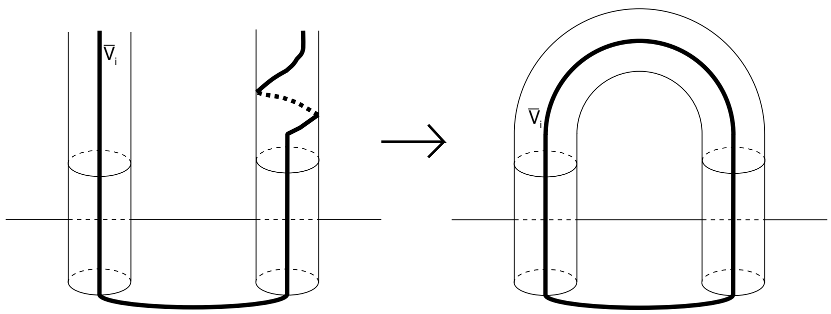


Figure 11: A type II move trivializing a handle.

4 Reducing to bottom stage of genus 1

One of the key moves we need we haven't even defined yet, the so-called "In-Out trick." This is because it only works for gropes that have bottom stage of genus 1. It therefore makes sense to first reduce the problem to that case.

4.1 The graph $\tilde{\Gamma}$.

Given an embedded grope G, we define a slightly different version of the graph defined in section 2.1. Fix a projection of the handlebody where all the $\overline{v}_i \cap B$ occur in increasing order as height decreases. For the graph, $\tilde{\Gamma}$,we let there be vertices v_i for every tip of the grope, as opposed to one for each of the n collections of tips with the trivialization property. We put an edge from a vertex to itself if that tip is framed linked in the previously defined sense (since it is just one tip you might say *framed knotted* instead), and we draw an edge between two vertices if the corresponding handles cross in the wrong order in the projection, as before.

In terms of the graph $\tilde{\Gamma}$ we can still do type $\tilde{I}$ and type $\tilde{II}$ moves, defined in the obvious analogous way. However, as analyzed at the beginning of section 2, the result of doing a type $\tilde{II}$ move is no longer neccessarily to trivialize the knot but instead to reduce the total genus of the grope, where total genus is defined as the sum of the genera of all the stages of the grope.

Lemma 4.1 *Let G be an embedded grope. If the graph $\tilde{\Gamma}(G)$ has a free set of k vertices then for any type $k-1$ invariant ν_{k-1}, we have that $\nu_{k-1}(\partial G) = \sum_i \pm \nu_{k-1}(\partial G_i)$, where G_i is an embedded grope of lower total genus than G, but of the same class.*

[Proof]

Let S be the scheme of type $\tilde{II}$ moves defined above. Then $\sum_{\sigma \subset S}(-1)^{|\sigma|}\nu_{k-1}(\partial G_\sigma) = 0$. If $\sigma \neq \emptyset$, then G modified by σ is of lower total genus as we just observed. $\Box$.

4.2 Genus 1 is sufficient

Lemma 4.2 *If theorem 1.1 holds for embedded gropes with bottom stage of genus* 1*, then it holds in general.*

[Proof]

Let $E(\tilde{\Gamma})$ be the number of edges. Consider, toward a contradiction, a counterexample which has minimal (total genus, $E(\tilde{\Gamma})$), ordered lexicographically. This example has bottom stage genus > 1, by assumption. Notice that $\tilde{\Gamma}$ has at least $2n$ vertices, since for each pair of dual symplectic basis elements in the bottom stage we get at least n vertices. I claim that $E(\tilde{\Gamma}) \leq \lceil \frac{n}{2} \rceil$. Otherwise, consider a scheme, S, consisting of $\lceil \frac{n}{2} \rceil + 1$ type $\tilde{I}$ moves.

If ν is a type $\lceil \frac{n}{2} \rceil$ invariant, we have,

$$\nu(K) = -\sum_{\emptyset \neq \sigma \subset S} (-1)^{|\sigma|} \nu(K_\sigma) \tag{1}$$

where each of the K_σ on the right-hand side has fewer edges but equal total genus. Hence each of these knots has reduced complexity, so that, by minimality each is $\lceil \frac{n}{2} \rceil$-trivial. That, of course implies that $\nu(K_\sigma) = 0$ for each $\sigma \neq \emptyset$, and yields the equation $\nu(K) = 0$, for every ν of type $\lceil \frac{n}{2} \rceil$, contradicting that K is a counterexample.

So $E = E(\tilde{\Gamma}) \leq \lceil \frac{n}{2} \rceil$. Recall that we argued that the number of vertices V exceeds or equals $2n$.

Let b_0, b_1 be the first two Betti numbers of the graph $\tilde{\Gamma}$. Then $b_0 - b_1 = \chi = V - E \geq 2n - \lceil \frac{n}{2} \rceil \geq \lceil \frac{n}{2} \rceil + 1$.

This shows there are at least $\lceil \frac{n}{2} \rceil + 1$ contractible components of Γ. In particular these components have no loops beginning and ending at the same vertex. Hence we can choose a free set of $\lceil \frac{n}{2} \rceil + 1$ vertices by selecting one vertex from each of these components. So by lemma 4.1, for every type $\lceil \frac{n}{2} \rceil$ invariant ν, $\nu(\partial G) = \sum \pm \nu(\partial G_i) = 0$ since each G_i is of lower total genus. But this is a contradiction, since K was supposed to be a counterexample. □

5 The In-Out trick (The hard move)

Let G be an embedded grope whose bottom stage is of genus one. Then G divides naturally into two halves: the two embedded subgropes attached to the bottom stage. A vertex of the associated graph is a set of tips which lies on exactly one of these two halves. Given a vertex, X, let G_X denote the half of G on which lie the tips in X. Suppose X is framed unlinked. The aim of this section is to construct a pair of moves, $\{\text{in}X, \text{out}X\}$ which have the following effect, at least when not done in conjunction with other moves.

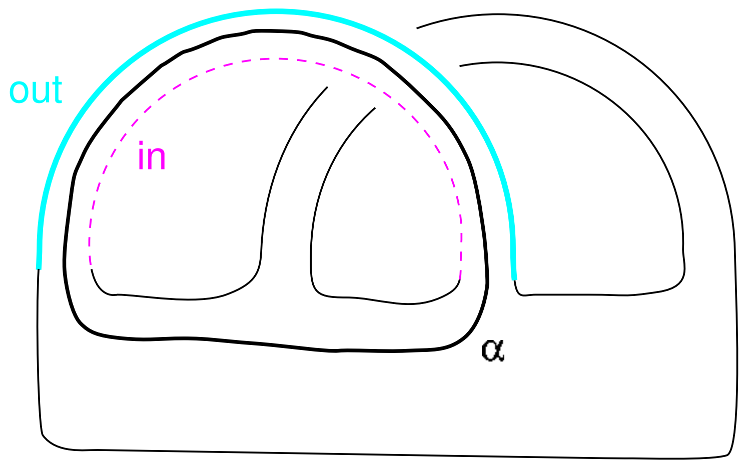


Figure 12: The "in" and "out" arcs.

1) in$X$ takes ∂G to ∂G_X.

2) out$X$ takes ∂G to $\rho(\partial G_X)$.[3]

3) $\{\text{in}X, \text{out}X\}$ takes ∂G to the unknot.

The two moves of the in-out trick are each divided into four phases. The expression in$X$ or out$X$ will refer to just doing phases I and II. Phases III and IV are isotopies which may or may not be possible if one is doing other moves in conjunction with the in-out pair. This will be analyzed on a case-by-case basis as the need arises.

We proceed to describe the moves. Suppose $X = \{x_1, \ldots, x_m\}$. Recall that each x_i is a tip in X. Let $\Delta_{x_1}, \ldots, \Delta_{x_m}$ be a cap. That is, $\{\Delta_{x_i}\}$ are disks such that $\cup\Delta_{x_i}$ is embedded, $\partial(\cup\Delta_{x_i}) = \overline{x}_1 \cup \cdots \cup \overline{x}_m$, and such that the disks' interiors may only intersect the handlebody at handles not associated to the the tips in X. Define two subarcs of ∂G called "in" and "out" as in figure 12, where $\alpha = \partial G_X$.

If a handle H intersects int$\Delta_{x_{i_0}}$, choose an arc inside $\Delta_{x_{i_0}}$ from $H \cap \text{int}\Delta_{x_{i_0}}$ to $\overline{x}_{i_0}$, terminating on the handle associated to x_{i_0}. Do this for all such intersections of a handle with the interior of the cap of X, choosing the arcs to all be disjoint.

We now describe a homotopy of G which does not preserve embeddedness but which restricts to an isotopy of ∂G. The arcs we selected end on handles associated to X. The cross-section of G inside the handle looks like some Ξ_i at that point along the handle. Push each handle, H, hitting the interior of the cap down the arc we chose until its intersection with the cap has been traded for two intersections with the top stage of G. See figure 13.

Continue pushing each handle down through the successive stages of G in a small

[3] ρ is the map which reverses a knot's orientation. It is unknown whether finite type invariants can detect a knot's orientation.

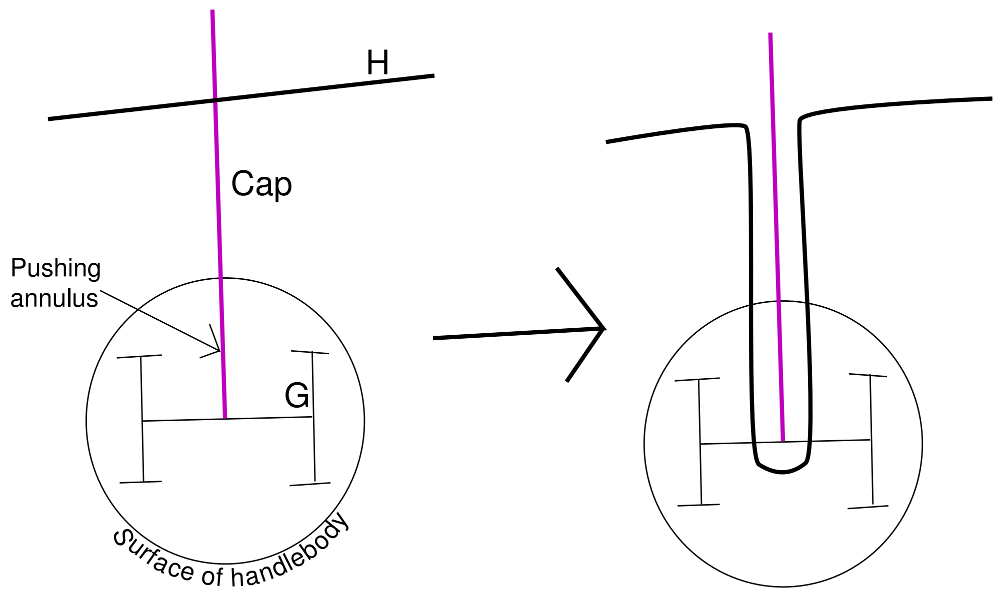


Figure 13: The first step of the in-out trick.

neighborhood of the cross-section until all the intersections of each handle are with the bottom stage of G. See figure 14.

Notice at this point that the knot ∂G never crossed itself, so that our procedure was just an isotopy of ∂G. This preliminary isotopy will be called *phase I* of the in-out trick.

Phase I introduces many intersections with the bottom stage, and these are naturally paired together. In figure 15, a typical such pair of intersections of a handle H with the bottom stage of G is pictured. Here $K = \partial G$. Define the moves $\text{in}X, \text{out}X$ by doing the illustrated move for every such pair of intersections with the bottom stage. $\text{in}X$ and $\text{out}X$ are clearly disjoint homotopies after phase I. Doing either $\text{in}X$ or $\text{out}X$ is called *phase II.*

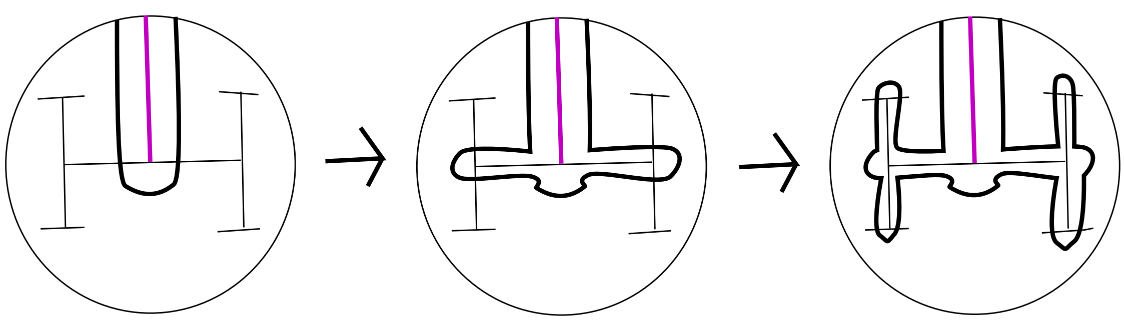


Figure 14: Pushing H down so that all intersections are with the bottom stage.

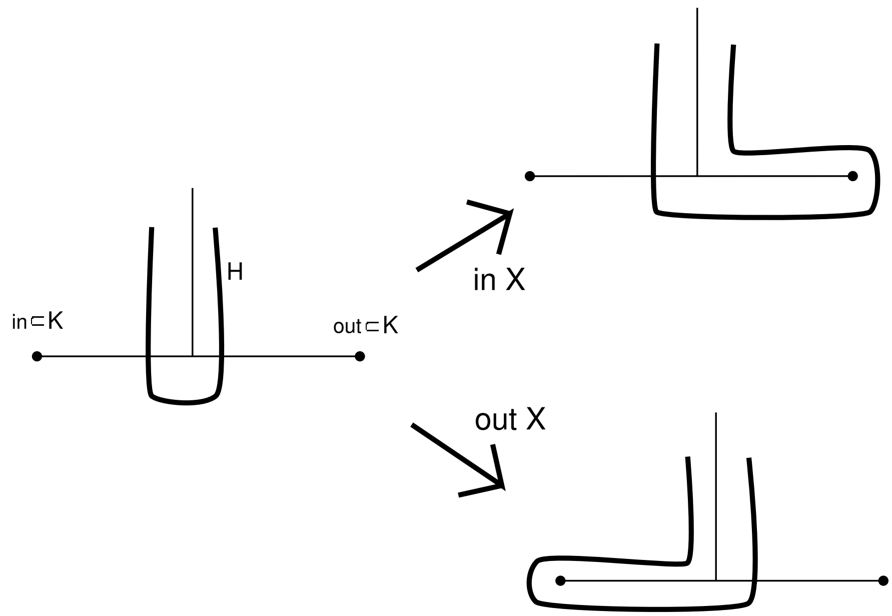


Figure 15: The "in" and "out" moves.

We now analyze what happens when we do in$X$, out$X$ or both. Doing both is the easiest case to tackle since it yields an embedded grope with the tips in X bounding the embedded disks $\{\Delta_{x_i}\}$. Doing both in$X$ and out$X$ therefore turns ∂G into an unknot.

Recall the curve $\alpha = \partial G_X$ of figure 12. Let G_X modified by phase I be called G'_X. G'_X is embedded in the sense that it does not intersect itself, but it may intersect the bottom stage of G if a handle that we pushed down comes from the G_X half of G. We can turn G'_X into a disk Δ by the construction at the beginning of section 2 in a regular neighborhood of $G'_X \cup \bigcup_i \Delta_{x_i}$. This disk may intersect the bottom stage of G if G'_X does. Now consider what happens when we do in$X$. All intersections of handles between the "in" arc and $\alpha = \partial\Delta$ have been removed. Consider the arc μ defined in figure 16. The closed curve gotten by joining the endpoints of "in" and μ in the obvious way cobounds an annulus with α. α bounds the disk Δ in the complement of this annulus and of the knot. It follows that the "in" arc is isotopic to μ in the complement of the rest of the knot. This isotopy to μ is called *Phase III.* The "out" arc was never made to cross itself, so after the "in" arc moves to μ, the "out" arc can be isotoped back to its original position. But now the band dual to α pulls away, and we are left with $\alpha = \partial G'_X = \partial G_X$. See figure 16. This final isotopy is *phase IV.*

A similar analysis holds for doing out$X$, but one must pay attention to orientations. If α is oriented the same way as the "in" arc, then it will be oriented oppositely to the "out" arc. Hence after doing out$X$ we get the knot $\rho(\partial G'_X)$.

For a genus one surface, the "in" and "out" arcs are symmetric so the move outX

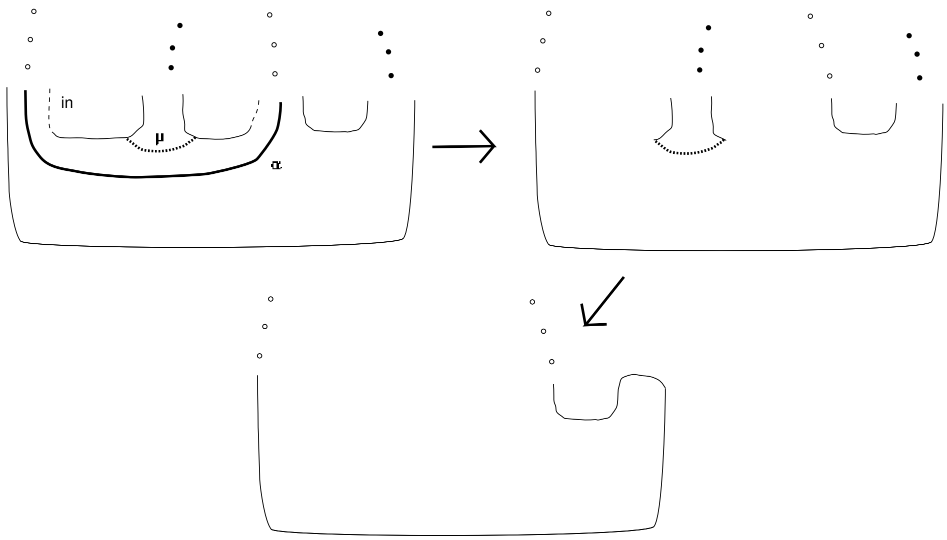


Figure 16: The move in$X$ gives the knot α bounding the grope G_X.

gives the same (unoriented) result as inX. However, for a higher genus surface, the "out" move no longer works, the problem occurring during phase IV, which is why we need the bottom stage of G to be genus one.

We now give an example. Consider the knot bounding an embedded grope of class 3 pictured in the upper left-hand corner of figure 17. We will describe the in and out moves on v_2. Notice that the "in" and "out" arcs have been indicated in the picture. The bold box indicates the region in which we will be doing the homotopy. Indeed the moves will be described by substituting the pictures at the bottom of figure 17 into the box. There is an obvious cap for v_2, which is pictured in the upper right-hand corner of figure 17. Doing phase I guided by the pictured arc is shown in the middle of the picture. Finally, inv_2, outv_2 and $\{\text{in}v_2, \text{out}v_2\}$ are pictured at the bottom of the figure.

In this case the curve "α" is unknotted, so that doing any combination of the in and out moves ought to produce the unknot, which we invite the reader to verify.

5.1 A fun proof that all knots are 1-trivial

We now use the in-out trick to give a proof that every knot is 1-trivial. This also follows from the main theorem and is well-known, but is good for illustrative purposes.

Suppose a knot, K, bounds a Seifert surface with k pairs of dual bands $\{x_i, y_i\}_{i=1}^k$. Consider the scheme $S = \{s_1, s_2\}$ where s_1 is the move which unknots and untwists the x_1 band and also does crossing changes with other bands so that x_1 always crosses over them. s_2 does a similar thing for y_1. Doing either s_1 or s_2 reduces the genus of the Seifert surface and hence type 1 invariants vanish inductively. Doing both gives a

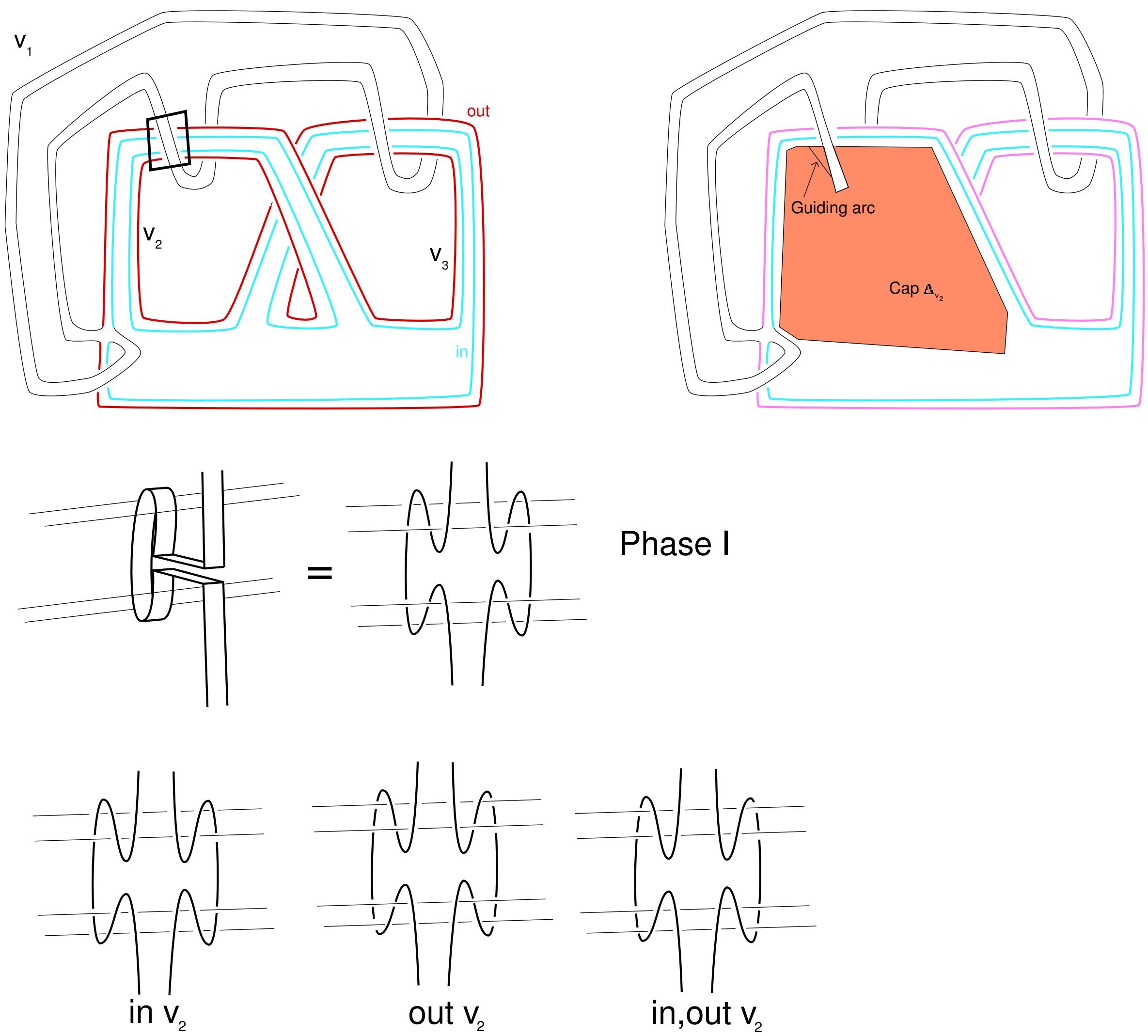


Figure 17: An example of “in” and “out” in action.

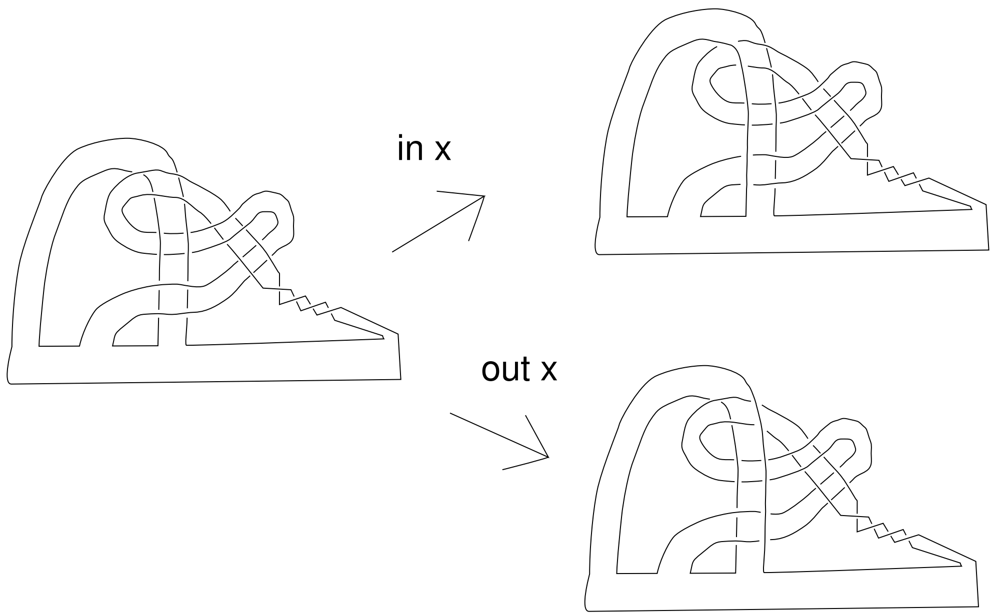


Figure 18: The knots on the right are unknots.

connected sum of a genus one knot that has unknotted bands with a reduced genus knot. 1 triviality is easily seen to be preserved by connected sum. Thus it suffices to prove that a genus one knot with unknotted bands, x, y, is 1-trivial. But every subset of moves of the scheme $\{\text{in}x, \text{out}x\}$ now trivializes the knot. In this simple case, in$x$ (respectively out$x$) may be visualized as the move making the "in" arc (respectively "out" arc) cross over everything in the projection. See figure 18.

5.2 Doing two in-out tricks

The following section is not needed to prove the main theorem, so the reader may skip there immediately. It is needed for the construction of the examples of the last section, however.

One might ask what happens when one does two of these in-out tricks simultaneously. That is one considers the scheme $\{\text{in}x, \text{out}x, \text{in}y, \text{out}y\}$ for two framed unlinked vertices x and y. Under the hypotheses of lemma 5.1, the answer is reasonably nice. If x and y lie on the same half of the grope, the answer is not so nice. The problem being that mixed terms like in$x$, out$y$ conflict with each other: after phase I of each move, intersections remain between the arc we called α and the "in" arc and between α and the "out" arc.

Lemma 5.1 *Consider an embedded grope G with genus one bottom stage which is formed by gluing the embedded gropes G' and G'' to the bottom stage. They intersect in a point, $*$. See figure 19. There are two ways to resolve this intersection of $\partial G'$ and $\partial G''$ inside*

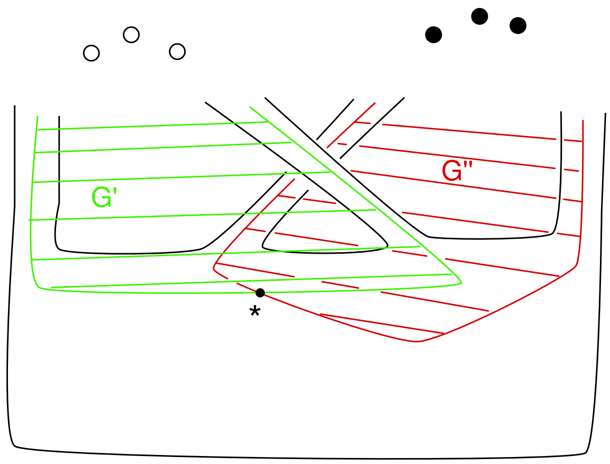


Figure 19: The local picture at the bottom stage of G.

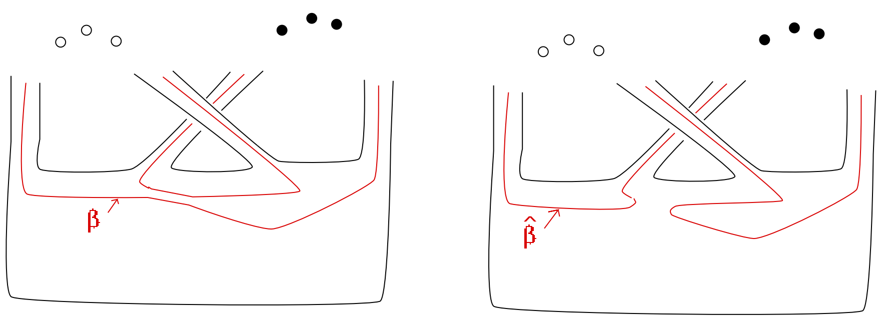


Figure 20: The two resolutions.

the bottom stage as pictured in figure 20. These give rise to two knots which are denoted β and $\hat{\beta}$, for some choice of orientation. Let x be a framed unlinked vertex on the G' half and y a framed unlinked vertex on the G'' half such that $\{x, y\}$ is not an edge in Γ. Consider the scheme $S = \{\mathrm{in}x, \mathrm{out}x, \mathrm{in}y, \mathrm{out}y\}$. Then

$$\sum_{\sigma \subset S} (-1)^{|\sigma|} \partial G_\sigma = \partial G + \beta + \hat{\beta} + \rho(\beta) + \rho(\hat{\beta})$$
$$-2(\partial G' + \rho(\partial G') + \partial G'' + \rho(\partial G'')) + 3\mathrm{unknot} \in \mathbb{Z}\mathrm{Knots}.$$

[Proof]

Consider figure 19 depicting a neighborhood of $G' \cap G''$. Recall that in the in-out trick an arc μ was defined. We needed to define one such arc μ for both the "in" move and the "out" move. Hence in this case there are four μ arcs $\mu_1, \ldots, \mu_4$, which we picture in

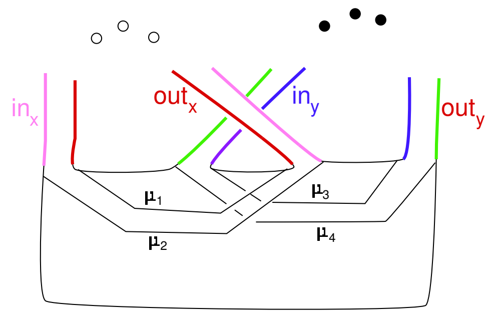


Figure 21: The "in", "out" and μ arcs.

figure 21. We have pictured the arcs slightly perturbed off of the surface so that they don't intersect. μ_1 and μ_2 have been perturbed along G' whereas μ_3 and μ_4 have been perturbed along G''. We have also labelled the arcs $\text{in}_x, \text{out}_x$ and $\text{in}_y, \text{out}_y$ in the figure. The four "in" and "out" arcs each have a μ arc connecting the endpoints. The claim is that to perform a combination of moves in our scheme S, one simply replaces the appropriate "in" and "out" arcs by the corresponding μ arcs. For instance, if one were to do in$x$ and out$y$ one would replace the in_x arc by μ_1 and the out_y arc by μ_4.

Let G'_I (resp. G''_I) be G' (resp. G'') modified by stage I of the in-out trick on x (resp. y). Both phase II and III of the in and out move on x are supported in a regular neighborhood of G'_I, whereas phases II and III of the in and out moves on y are supported in a regular neighborhood of G''_I. These two open sets intersect in a small ball around $*$. Since we perturbed the μ arcs out along G'_I and G''_I, away from $*$, one sees that the phase III isotopies are disjoint. Hence the in and out x moves up to phase III will be disjoint from the in and out y moves up to phase III. The last thing to check is that two moves on the same vertex are disjoint up to phase III. This is readily checked: if one does say in and out on x, first of all we already know the moves are disjoint up to phase II. Then there is a disk Δ which we defined which provides an isotopy of the arc "in" with μ_1 and an isotopy of the arc "out" with μ_2. These isotopies are easily made disjoint by using two parallel copies of Δ.

Consider figure 22.

It depicts doing all subsets of the scheme S. When one passes through a domain wall marked by a homotopy, one does the homotopy. Hence, for instance the right hand part of the diagram is the the left-hand part, with the move in$y$ also done. One can choose orientations for $\partial G', \partial G'', \beta, \hat{\beta}$ arbitrarily, although to be consistent with our earlier convention in$x$ should give $\partial G'$ and in$y$ should give $\partial G''$. The lemma now follows by taking the signed sum of all terms in figure 22. □

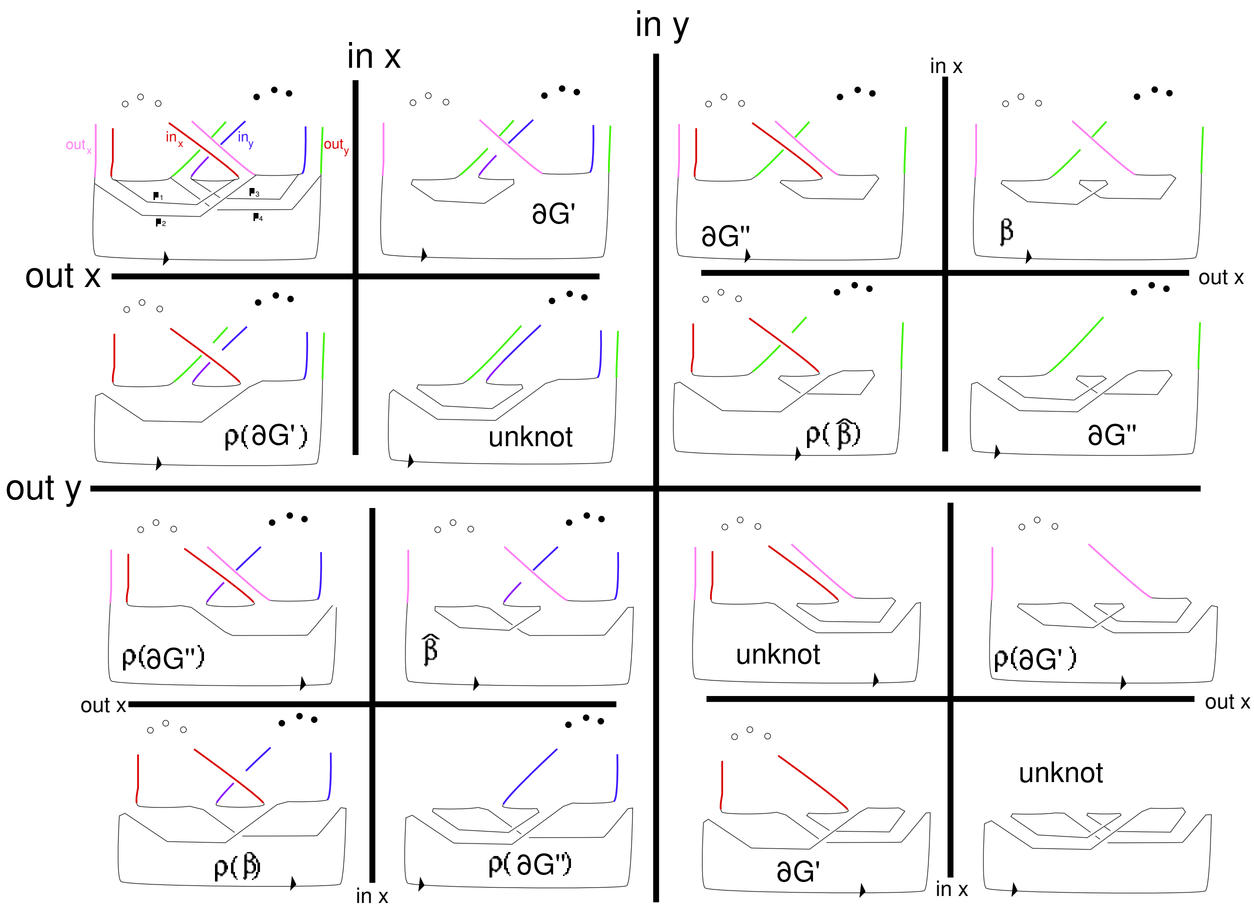


Figure 22: The scheme S.

6 Proof of theorem 1.1

Call an embedded grope of class $2m+1$, which has bottom stage of genus 1 and whose associated graph has at most $m+1$ edges, *m-atomic*.

Lemma 6.1 *If theorem 1.1 holds for m-atomic embedded gropes, then it holds in general.*

[Proof]

The fact that we need only consider embedded gropes whose bottom stage is genus 1 is lemma 4.1. We may assume $n = 2m+1$ since the even case follows by thinking of a class $2m$ grope as a class $2m-1$ grope by forgetting a stage and $\lceil \frac{2m-1}{2} \rceil = \lceil \frac{2m}{2} \} \rceil$. Also, we may assume $E(\Gamma) \leq m+1$, using an argument like the proof of lemma 4.1, since we have $m+2$ moves in hand to reduce the number of edges. $\Box$

Note that the associated graph to any m-atomic embedded grope has a free set of m vertices by a simple Euler characteristic argument like the one given to prove lemma 4.1.

Lemma 6.2 *Suppose theorem 1.1 holds for every m-atomic embedded grope for which the associated graph has the property that the complement of any free set of m vertices is free. Then theorem 1.1 holds for all m-atomic embedded gropes.*

[Proof]

Suppose not. Let G be a counterexample with minimal $E(\Gamma\backslash\text{star}F)$, where $F = \{v_1, \ldots, v_m\}$ is a free set of m vertices. Being a counterexample, it is not $m+1$-trivial. Let $S = \{s_1, \ldots, s_{m+2}\}$ be the scheme in which $s_1, \ldots, s_{m-1}$ are type II moves trivializing the $V_1, \ldots, V_{m-1}$ handles supported between separating planes. s_m, s_{m+1} are the in and out move respectively on the V_m handles. These two moves are supported in a neighborhood of the V_m handles with caps, which is separated from the $V_1, \ldots, V_{m-1}$ handles by hyperplanes, and so is disjointly supported from the type II moves. Finally, s_{m+2} is a type I move which reduces $E(\Gamma\backslash\text{star}F)$. It is possible that a regular neighborhood of the support of s_{m+2}, $N(\text{supp}(s_{m+2})) \cong \amalg D^3$ is not disjoint from the type II moves: a type II move might pull the handles it is contracting through one of these balls. Since s_{m+2} is deleting an edge away from $V_1, \ldots, V_{m-1}$, at least the handles that are being contracted by the type II moves don't start out hitting the balls. One then chooses the separating planes for the type II moves so they don't hit the balls and chooses the moves themselves to avoid the balls. Finally, s_{m+2} is clearly disjoint from the in and out moves using that s_{m+2} is deleting an edge that doesn't hit the vertex V_m.

So for any type $m+1$ invariant ν_{m+1}, $\sum_{\sigma \subset S} (-1)^{|\sigma|} \nu_{m+1}(\partial G) = 0$, and let us see what this says. To prepare, let us suppose that G is formed by attaching the embedded gropes H' and H'' to the dual bands of the bottom stage, thereby partitioning the set of vertices of $\Gamma(G)$, $\mathcal{V}$ into two nonempty sets $\mathcal{V}_{H'}$ and $\mathcal{V}_{H''}$. Suppose without loss that $V_m \in \mathcal{V}_{H'}$. Let $S_{H'}$ and $S_{H''}$ partition $\{s_1, \ldots, s_{m-1}\}$ into two sets in the obvious way. Let $S_I = \{s_m, s_{m+1}\}$ and $S_C = \{s_{m+2}\}$.

Note that we can assume s_{m+2} reduces $E(\Gamma\backslash\mathcal{V}_{H'})$ since if this were zero, then $\mathcal{V}_{H''}$ would have no edges hitting it. By the assumption that the height function separates the two halves of the embedded grope H' and H'' (see section 2.1), the handles on the H'' half all bound disks, implying of course that the grope is trivial, contradicting the premise that G is a counterexample. Thus we can assume some complexity not contained wholly within the H' half, and without loss s_{m+2} reduces this.

We are now in a position to describe what happens under the various combinations of moves from $S_{H'}, S_{H''}, S_I$ and S_C, with the initial assumption that neither $S_{H'}$ nor $S_{H''}$ is empty. For easy reference, here is a table describing the four subsets of S:

$S_{H'}$:	type II moves on H' handles
$S_{H''}$:	type II moves on H'' handles
S_I :	in and out move on a handle in H'
S_C :	type I move reducing $c(\Gamma\backslash\mathcal{V}_{H'})$

In the following list of cases, case i refers to a set of moves, σ, which hits i of the above 4 sets.

Case 0

This is the empty move yielding $\boldsymbol{\partial G}$.

Case 1

By our previous analysis of the handle-trivializing moves, if $\sigma \subset S_{H'}$ or $\sigma \subset S_{H''}$, ∂G_σ is the **unknot**. $\partial G_{s_{m+2}}$ has fewer of the appropriate edges so by minimality $\nu_{m+1}(\partial G_{s_{m+2}}) = \mathbf{0}$. The leftover terms are the ones gotten from the in-out trick: doing both of s_m, s_{m+1} is the **unknot**, while $\partial G_{s_m}, \partial G_{s_{m+1}}$ are $\boldsymbol{\partial H'}$ and $\boldsymbol{\rho(\partial H')}$.

Case 2

σ hits $S_{H'}, S_{H''}$: **unknot**.

σ hits $S_{H'}, S_I$: $S_{H'}$ trivializes some handles, and then s_m or s_{m+1} give H' with trivialized handles, an **unknot**. Doing both the in and out move also yields an **unknot**.

σ hits $S_{H'}, S_C$: $S_{H'}$ trivializes handles of the embedded grope $G_{s_{m+2}}$ yielding an **unknot**.

σ hits $S_{H''}, S_I$: $S_{H''}$ gives some embedded grope with the H' half unaltered. Doing one move from S_I then gives the H' half. Recording all of these, we get

$$\sum_{\emptyset\neq\tau\subset S_{H''}} (-1)^{|\tau|+1}\{\nu_{m+1}(\partial H') + \nu_{m+1}(\rho(\partial H'))\}.$$

Again if we do both s_m and s_{m+1} the result is obviously an **unknot.**

σ hits $S_{H''}, S_C$: **unknot**.

σ hits S_I, S_C: S_C gives some embedded grope with the H' half unaffected. So as in case 1 we get $\boldsymbol{\partial H'}$ and $\boldsymbol{\rho(\partial H')}$.

Case 3

σ hits $S_{H''}, S_I, S_C$: $S_{H''}, S_C$ give an embedded grope with H' half intact, and so as in case $2(S_{H''}, S_I)$ we get, adjusting the sign to include the s_{m+2} move,

$$\sum_{\emptyset \neq \tau \subset S_{H''}} (-1)^{|\tau|} \{\nu_{m+1}(\partial H') + \nu_{m+1}(\rho(\partial H'))\}.$$

σ hits $S_{H'}, S_I, S_C$: **unknot**. This is case $2(S_{H'}, S_I)$ applied to $G_{s_{m+2}}$.

σ hits $S_{H'}, S_{H''}, S_C$: **unknot**. This is case $2(S_{H'}, S_{H''})$ applied to $G_{s_{m+2}}$.

σ hits $S_{H'}, S_{H''}, S_I$: **unknot**. This is case $2(S_{H'}, S_I)$ applied to G_σ for $\sigma \subset S_{H''}$.

Case 4

This involves doing at least one move from each collection and is an **unknot**. This is case $3(S_{H'}, S_{H''}, S_I)$ applied to $G_{s_{m+2}}$.

We conclude

$$\sum_{\sigma \subset S} (-1)^{|\sigma|} \nu_{m+1}(\partial G_\sigma) = \nu_{m+1}(\partial G) - \nu_{m+1}(\partial H') - \nu_{m+1}(\rho(\partial H')) +$$

$$\sum_{\emptyset \neq \tau \subset S_{H''}} (-1)^{|\tau|+1} \{\nu_{m+1}(\partial H') + \nu_{m+1}(\rho(\partial H'))\}$$

$$+\nu_{m+1}(\partial H') + \nu_{m+1}(\rho(\partial H')) +$$

$$\sum_{\emptyset \neq \tau \subset S_{H''}} (-1)^{|\tau|} \{\nu_{m+1}(\partial H') + \nu_{m+1}(\rho(\partial H'))\} = \nu_{m+1}(\partial G) = 0$$

This contradicts that $\nu_{m+1}(\partial G) \neq 0$ for some type $m+1$ invariant.

If $S_{H'} = \emptyset$, then only cases leading to an $m+1$-trivial knot are eliminated so the calculation still goes through.

If $S_{H''} = \emptyset$, then two nontrivial cases are eliminated: the $S_{H''}, S_I$ subcase of case 2 and the $S_{H''}, S_I, S_C$ subcase of case 3. The calculation is now $\sum_{\sigma \subset S} (-1)^{|\sigma|} \nu_{m+1}(\partial G_\sigma) = \nu_{m+1}(\partial G) - \nu_{m+1}(\partial H') - \nu_{m+1}(\rho(\partial H')) + \nu_{m+1}(\partial H') + \nu_{m+1}(\rho(\partial H')) = 0$ which still yields the contradiction $\nu_{m+1}(\partial G) = 0$. □

Lemma 6.3 *Suppose a graph with $2m+1$ vertices has the property that the complement of every free set of m vertices is free, and furthermore that a free set of m vertices exists. Then the graph has no edges.*

[Proof]

Easy. □

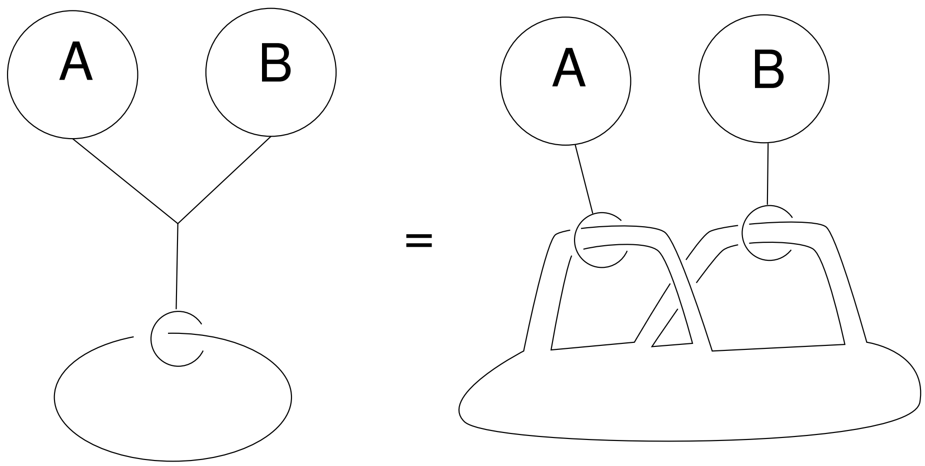


Figure 23: Habiro's move 9.

The proof of theorem 1.1 is now complete: the previous three lemmas imply that we need only check it for embedded gropes whose associated graph has no edges. Since the boundary of such an embedded grope is the unknot, theorem 1.1 is obvious in this case.

7 Showing the result is sharp

It is the purpose of this section to prove theorem 1.2. That is, to construct embedded gropes of class n which are not $\lceil \frac{n}{2} \rceil + 1$ trivial. In fact we construct embedded gropes of class $2m$ which are not $m+1$ trivial. These also serve as optimality examples for the odd case, as a knot bounding an embedded class $2m$ grope will bound a class $2m-1$ embedded grope by deleting a top stage. The deleted gropes still serve as examples because $\lceil \frac{2m-2}{2} \rceil + 1 = m+1$.

The examples are constructed using graph clasper calculus [H2]. As needed ingredients we obtain some formulas (equations (2),(3) and lemma 7.2) and a result on link triviality (lemma 7.1) we hope are of independent interest. The invariants we use to detect non-triviality of our examples are Jones polynomial coefficients, whose weight system was essentially derived by Bar-Natan [B-N].

To begin, we briefly explain why certain claspers yield gropes. This is explained in more detail in [CT]. Consider figure 23. Here A and B are both tree claspers with no boxes with n and m respectively non-root leaves. The shown equality is move 9 of [H2]. Inductively, A ties a grope of class n into a band of the depicted surface, and B ties a grope of class m into the dual band. The result is a grope of class $n+m$.

Our examples are depicted in figure 24. The two versions are equal by move 2 of [H2]. Surgery on the given clasper yields an embedded grope of class $2m$, which can be seen

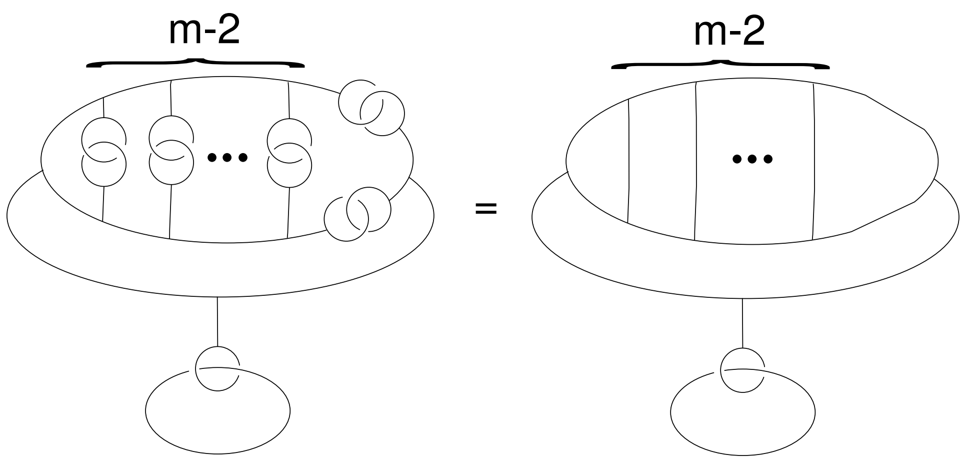
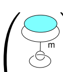
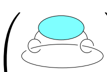


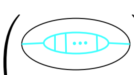
Figure 24: The examples.

by counting the tips (non-root leaves). Denote this embedded grope by G^{2m}.

Suppose the coefficients of the $t = e^x$ Taylor expansion of the Jones polynomial are denoted by $\{j_k\}$. We will show that $j_{m+1}(\partial G^{2m}) = 2^{m+3} - 2^{m+1} \neq 0$, demonstrating the optimality of theorem 1.1. We will establish the following formula on graph claspers.

$j_{m+1}\Bigg($  $\Bigg) = 2 j_m \Bigg($  $\Bigg),$ (2)

where the claspers are of degree m. Notice that the right hand side of this formula involves a type m invariant evaluating on a degree m graph clasper. By section 8.2 of [H2], this is the same as evaluating the type m invariant on the corresponding chinese character diagram.[4] Our result will follow from

$j_m\Bigg($  $\Bigg) = 2^{m+2} - 2^m$ (3)

which will follow from Bar-Natan's derivation of the HOMFLY polynomial weight systems.

7.1 Link triviality

In this section we analyze a phenomenon noticed by Habiro([H2], prop. 7.4), which is that certain clasper surgeries preserve finite type invariants of a much larger degree than expected. We will only use a special case of lemma 7.1, namely for a graph clasper of

[4] Actually, Habiro only announces this result without proof. The author has verified the claim.

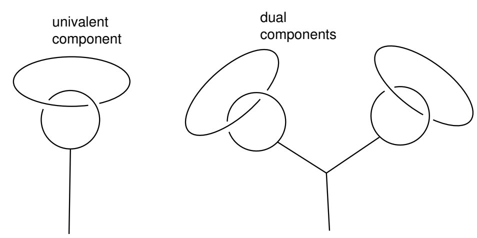


Figure 25: Univalent and dual components.

the same form as L in figure 30. The general result is sufficiently interesting to merit inclusion. The phenomenon we are studying is already present in the case of a graph clasper representing a grope: it preserves invariants of the clasper's degree, which is one higher than expected. This case is, of course, easy since the corresponding chinese character diagram is trivial modulo STU. The essential feature of this example is that the clasper has only one leaf which hits the knot.

Definition 7.1 *a) Let C be a graph clasper on the unlink U_k. If a component of U_k only hits one disk leaf of the clasper as in the left-hand side of figure 25, then we call that component* univalent.

b) Two univalent components are said to be dual *if the disk leaves that hit them have emanating edges which meet at a vertex. See figure 25.*

Lemma 7.1 *Suppose C^d is a graph clasper of degree d on U_k, for which there are u univalent components, no two of which are dual. Then surgery on C^d preserves type $d + u - 1$ link invariants.*

Note: This lemma is optimal in the sense that there are graph claspers satisfying the hypotheses which don't preserve type $d + u$ invariants. Intriguingly, the claspers Habiro considers in prop 7.4 preserve one further degree than what lemma 7.1 claims.

[Proof]

First, cut apart C^d using move 2 of [H2] into a link of tree claspers C_i such that:

(i) Each C_i has a designated root, which is a disk leaf hitting a component of U_k.

(ii) Every univalent component of U_k hits a root.

(iii) Each clasper is at least of degree 2.

An example is pictured in figure 26. For now, ignore the stars and the "in,out" markings.

Modify U_k via successive applications of the procedure depicted in figure 23 beginning at the root of each tree. We will define a scheme of cardinality $d+u$ on this new link, L, which is the link obtained by surgery on C^d. There is an obvious scheme of d moves: each

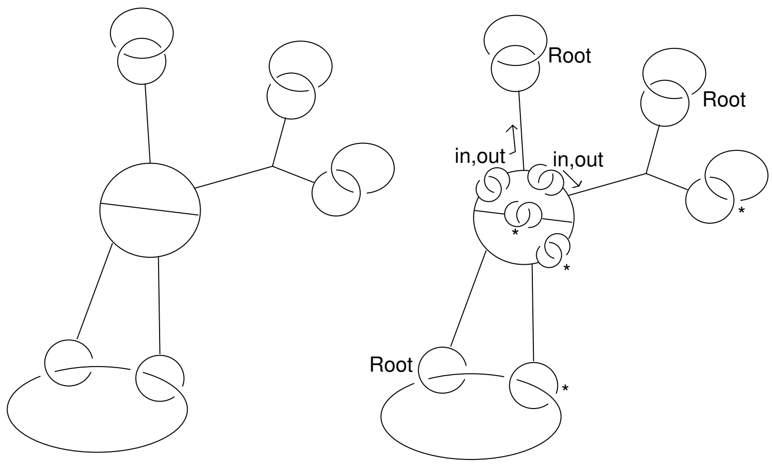


Figure 26: An example of lemma 7.1 Here $d = 6$, $k = 4$ and $b = 2$. Hence the pictured link is 7-trivial.

move either unlinks a hopf-linked pair of leaves, or pushes the link out of a non-root disk leaf. Call these types of moves *obvious* moves. (These moves are performed on L but can be viewed as being performed on the claspers.) We turn this scheme into a scheme of $d+u$ moves as follows. For each of the u claspers whose root hits a univalent component, choose a non-root leaf. Do this in such a way that no two such leaves are a hopf-linked pair. Each of these u claspers ties a grope into its univalent component, hence we can do an in-out pair of moves on each of these selected leaves. The new scheme is gotten by converting the u obvious moves associated to the u selected leaves to in-out pairs. In figure 26, we have marked obvious moves with an asterisk, and have also marked the in-out moves. The arrrows indicate which grope (clasper) we are doing the in-out pair on.

Now we have a scheme with the appropriate number of moves. The claim is that the usual alternating sum over the scheme is $L - U_k \in \mathbb{Z}Knots$. We prove this for a couple of illustrative examples including the one we need and leave the details of the general argument to the reader.

The in and out moves are depicted on the clasper level in figure 27.

The first example is given in figure 28. We have pictured the scheme we constructed at the beginning of this proof. In this case every nonempty subset of the scheme just gives the unlink U_2. This is because all moves will trivialize a leaf of at least one of the two claspers: hence that clasper just gives an isotopy and may be deleted. But once that clasper is deleted the other one will have a trivial leaf, and so may be deleted as well. Doing a typical subset of the scheme is shown in the right hand side of figure 28: the top in move and an obvious move have been done.

The second example is given at the top of figure 29. In this case the claim is that doing any nonempty subset of the in and out moves has the same effect whether or not

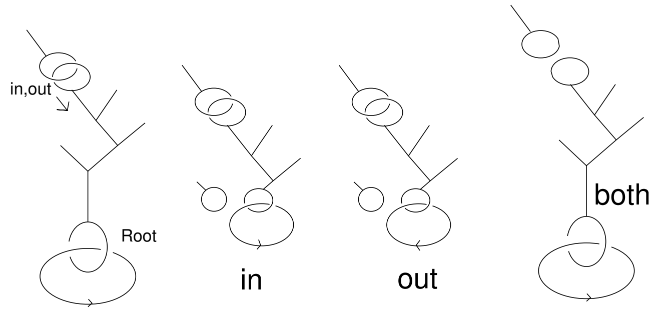


Figure 27: The in and out moves on the clasper level.

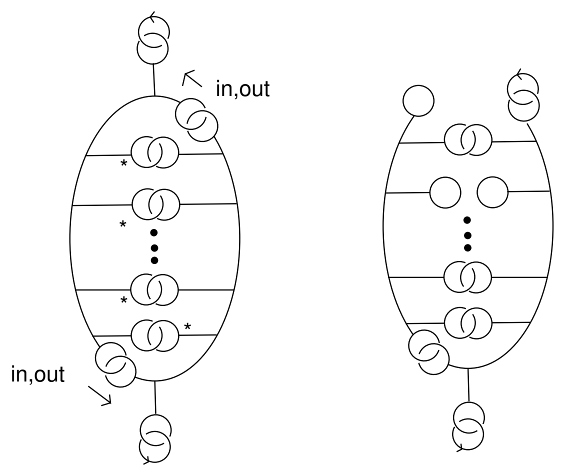


Figure 28: The first example of lemma 7.1.

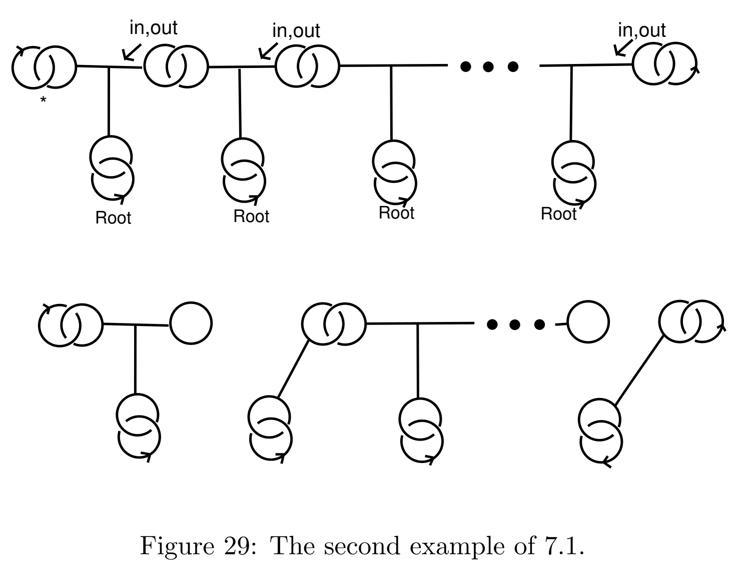


Figure 29: The second example of 7.1.

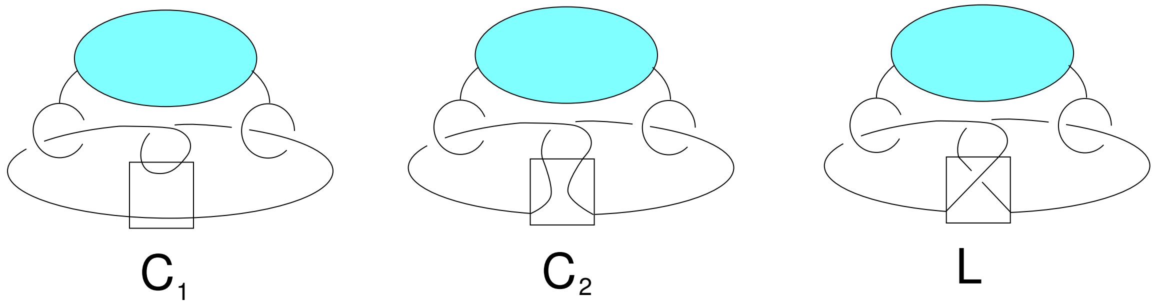


Figure 30: Three claspers corresponding to the Kaufmann bracket skein relation.

the one obvious move is done. This is illustrated for at the bottom of figure 29: an in and an out move are pictured, and one can see that unlinking the link component from the left-hand clasper makes no difference since the clasper can be deleted anyway. Therefore all terms involving the in and out moves cancel in pairs, and the alternating sum is evidently as stated. □

7.2 Claspers and the Jones polynomial

Consider the three claspers of degree m pictured in figure 30. Here C_1, C_2 and L refer to the link gotten by surgery along the pictured clasper. The skein relation for the Kaufmann bracket yields the equation

$$A^{-1} < C_1 > +A < C_2 >=< L > .$$

Let w denote the writhe. Assume it is zero away from the box enclosing the skein relation. Then $w(C_1) = 1$, $w(C_2) = -1$ and $w(L) = 0$. Recalling that the Jones polynomial J is given by $J_K = (-A)^{-3w} < K >$, and making the standard substitution $A^{-2} = e^{\frac{x}{2}}$, we achieve the following relation:

$$-e^{-\frac{x}{2}} J_{C_1}(x) - e^{\frac{x}{2}} J_{C_2}(x) = J_L(x) \tag{4}$$

Notice that C_1 and C_2 are degree m clasper surgeries on the unknot, hence they are $m-1$-trivial. On the other hand, by lemma 7.1, L is actually $m+1$ trivial. Let U_2 denote an unlink of two components. Thus

$$J_{C_1}(x) = 1 + j_m(C_1)x^m + j_{m+1}(C_1)x^{m+1} + o(x^{m+2})$$
$$J_{C_2}(x) = 1 + j_m(C_2)x^m + j_{m+1}(C_2)x^{m+1} + o(x^{m+2})$$
$$J_L(x) = j_0(U_2) + o(x^{m+2})$$

Recall that the Jones polynomial of an unlink of two components is $-e^{-\frac{x}{2}} - e^{\frac{x}{2}}$, and hence that $j_0 = -2$. Now, plugging these into equation 4 and comparing the coefficients

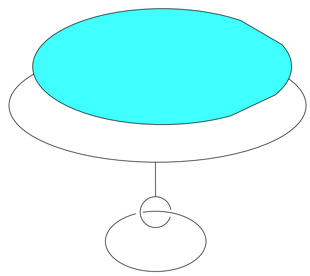


Figure 31: G is the unknot surgered along the pictured clasper.

of x^{m+1}, we get the following equation:

$$\frac{1}{2}j_m(C_1) - \frac{1}{2}j_m(C_2) - j_{m+1}(C_0) - j_{m+1}(C_1) = 0 \tag{5}$$

Notice that C_2 can be thought of as C_1 with an extra half-twist on one its edges. On the level of chinese character diagrams this corresponds to a switch in sign. Therefore $j_m(C_2) = -j_m(C_1)$, and equation 5 becomes

$$j_{m+1}(C_1) + j_{m+1}(C_2) = j_m(C_1) \tag{6}$$

7.3 An STU-like relation

In figure 31, a degree m clasper is defined, where the shaded region is the same as in C_0, C_1 and L.

We have the following lemma giving an STU-like relation at one degree higher than the claspers' degree.

Lemma 7.2 *Let ν_{m+1} be a type $m+1$ invariant. Then $\nu_{m+1}(G) = \nu_{m+1}(C_1)+\nu_{m+1}(C_2)+\nu_{m+1}(\rho(C_1)) + \nu_{m+1}(\rho(C_2))$.*[5]

[Proof]

The proof uses lemma 5.1. One considers G as a grope by breaking apart appropriate edges using move 2 of [H2]. There are m moves corresponding to unlinking these hopf-linked pairs of tips/leaves. Pick a pair that has one tip x belonging to one half of the grope, and another pair where there is a tip y belonging to the other half. We consider the scheme of $m+2$ moves gotten by doing the $m-2$ hopf-pair unlinking moves, and also the in and out moves on both x and y. The claim is that if one does any of the hopf pair

[5] Recall ρ is the map reversing the knot's orientation.

unlinkings, even in conjunction with the in-out moves, the result is unknotted. One can verify this by noting that doing a set of the in-out moves gives one of the "β" curves or one half of the grope. (In our setting, each half of the grope is unknotted in the absence of the other.) Now if one does some of the in-out moves, and further unlinks some hopf pairs of tips, one gets the analog of the "β" curves or grope halves, only with respect to the grope with the unlinked tips. But all these curves are the unknot. Hence the only nontrivial moves of the scheme correspond to subsets of the in-out moves. These are analyzed in lemma 5.1. It now simply remains to observe that in our case $\beta = C_1$ and $\hat{\beta} = C_2$. □

7.4 Bar-Natan's HOMFLY weight system and the rest of the proof

Lemma 7.2 gives us the equation, remembering that the Jones polynomial is knot orientation independent,

$$j_{m+1}(G) = 2\left(j_{m+1}(C_1) + j_{m+1}(C_2)\right)$$

which together with (6) implies that,

$$j_{m+1}(G) = 2j_m(C_1) \tag{7}$$

At this point we've shown that if, for some choice of filling in the shaded oval in C_1, $j_m(C_1) \neq 0$, then the embedded grope G of class $2m$ is not $m+1$-trivial as desired. As we stated earlier, we will do this for the choice given in equation 3.

Bar-Natan[B-N] derives a weight system corresponding to the Lie algebra $gl(N)$ which gives a multiple of the coefficients of the HOMFLY polynomial. Specifying $N = 2$, one gets a weight system corresponding to a multiple of the Jones coefficients. We reproduce the formula (p. 463, formula (36)) of [B-N]:

$$\mathcal{T}_2(D) = \sum_M (-1)^{s_M} 2^{b(\tau D_M)} \tag{8}$$

Here D is a connected Chinese character diagram.[6] M is a marking of the internal vertices of D by the digit 0 or 1. s_M is the sum of the digits of M. $b(\tau D_M)$ is the number of boundary components of the thickening $\tau(D_M)$ of D_M. The two ways of thickening a vertex are depicted in figure 32. Let $\mathcal{T}_2^i$ refer to the restriction of $\mathcal{T}_2$ to diagrams of degree i. If one works out the details of section 6.5 of [B-N], one discovers that

$$j_i = \frac{1}{2}\mathcal{T}_2^i.$$

The reason for the factor of $\frac{1}{2}$ is that $\mathcal{T}_2^0$ applied to an unknot is 2, whereas j_0 is 1 on an unknot.

[6]The formula as presented is actually for the framed Jones polynomial, and in this sense will work for non-connected diagrams. Deframing a weight system via exercise 3.22 of [B-N] does not affect connected diagrams.

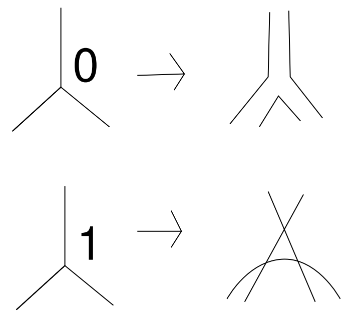
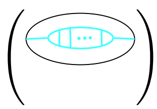


Figure 32: Thickening a marked diagram.

Lemma 7.3 *Suppose* $m \geq 2$. $\mathcal{T}_2^m$  $= 2^{m+2} - 2^m$, *where the depicted diagram has* $m-2$ *vertical struts.*

[Proof]

The statement is easily verified for the case $m = 2$. For $m > 2$, we consider the eight possible markings of the left-hand triangle. See figure 33. Here $\mathcal{T}_2^m$ is represented as a sum over the eight markings of the triangle, some of which have been consolidated due to symmetry. Also each yet-to-be-marked vertex has boxes surrounding it. If the vertex is marked with a 0, "=" signs are substituted for each surrounding box, and if the vertex is marked with a 1, the surrounding boxes are replaced by "X"s. We have pre-simplified some of the diagrams using the identity that 2 "X"s on the same edge cancel. The depicted equations are to be interpreted on the level of $\mathcal{T}_i^m$. Thus in the first row of figure 33 a factor of 2 pulls out due to the presence of an additional component of the thickening. The second row of terms is zero, because each term gives the same picture whether or not the x vertex is a 0 or a 1. Since these two possibilities have opposite sign, everything cancels. Finally the last row is zero, either by cancelling the two given terms or by noting that each is $\mathcal{T}_2^m$ applied to a diagram which is trivial modulo STU. The total is given in figure 34. If the diagram with $m-2$ struts is called D_m, then we have shown that $\mathcal{T}_2^m(D_m) = 2\mathcal{T}_2^{m-1}(D_{m-1})$. Hence we are done by induction. □

The alert reader will have noticed that we never constructed an example for the case $m = 1$. That is, an embedded grope of class 2 which is not 2-trivial. As all knots bound Seifert surfaces, this is the statement that there exist knots which are not 2-trivial. I believe that of the knots which have eight crossings or fewer, that 8_{20} is the only 2-trivial knot.

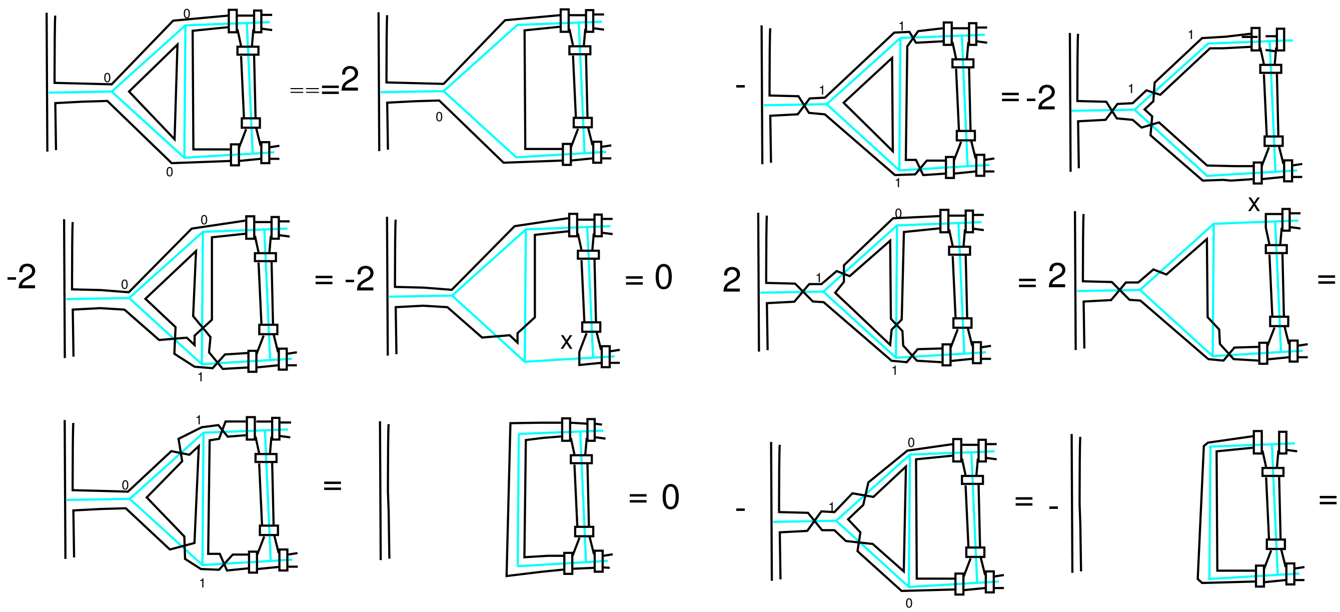


Figure 33:

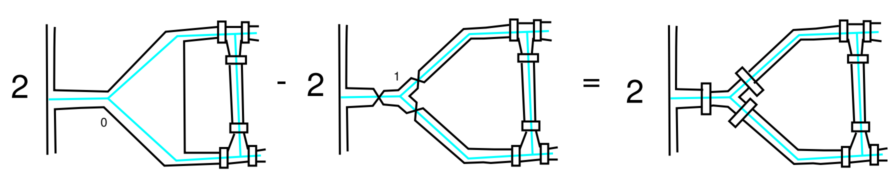


Figure 34:

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