— First worksheet on integrals, expanding on 7.1 of our textbook, with some morsels pertaining to chapters 7.2-7.4 already included; more to follow — Jochen Denzler Feb 2001

7.1: Basic Integration Formulas

To recognize the right method of finding antiderivatives (which also includes the capability of knowing when certain methods do *not* look promising and you better leave the integral alone) is kind of an art.

Some basic rules can be found in the textbook, and I will try to summarize them as succinctly as possible:

- If you encounter an integral under the headline “Basic Substitutions”, then you should try substitutions.
- If you encounter an integral under the headline “Completing the Square”, then you should try to complete squares.
- If you encounter an integral under the headline “Trigonometric Identities”, then you should try to use trigonometric identities.
- You will recognize the other rules in hte booke, so I needn’t repeat them: But there is also a final rule, which is only hidden between the lines of the textbook, and I will therfore spell it out explicitly:
- If you encounter an integral under the headline “Exam 2”, you turn in an empty sheet, go home and cry bitterly.

Just in case you do not like this approach, let me try to offer a more systematic one, which puts the single techniques in some context with certain goals in mind:

When you encounter an integral, then the information that is available to you is the expression under the integral alone. Certain strategies apply for certain types of integrands. If you have a really messy type of integrand that fails to fit in the categories, there is a good (bad) chance that the integral cannot be done in explicit formulas at all. The list of types of integrands that are amenable to your strategy toolkit will increase with experience. But the most important ones are already within your reach.

7.1.1: Rational Functions

If you prefer so, you can read the last paragraph of this section, namely the problems, first, and then return here to learn how you can do them.

Rational functions are functions that can be expressed in terms of the basic operations $+ \ - \ \times \ /$ alone: No roots, no trig functions, no logarithms, exponentials or whatever. The simplest examples are polynomials like $f(x) = x^5 + 3x^4 - \frac{1}{2}x + 17$. You know how to integrate them, so I won’t dwell on them. *If you don’t, you have a problem and should use the office hour urgently.*

Otherwise, all rational functions can be expressed as a quotient of two polynomials, like eg.

$$f_1(x) = \frac{x^2+3x+7}{2x^6+9x^4-x-1} \qquad \text{or} \qquad f_2(x) = \frac{7x^5-3x^2+x-6}{3x^3-x^2+4}$$

Clearly, you may encounter them in different disguises like

$$f_3(x) = \frac{\dfrac{1}{x^2+1} + \dfrac{x^2+1}{x^4+1} + x}{\dfrac{1}{x} + x^2 + \dfrac{x-7}{x+4}} \qquad \text{or like} \qquad f_4(x) = \frac{x}{x^2+x+1^2} + \frac{1}{x(x-2)} + \frac{x^3}{x-2}\,,$$

but you can always transform them into a quotient of polynomials. I am NOT saying, you should. Rather, the forms f_1 or f_2 are like a worst case szenario, and once you have understood how to handle them, you will find that other forms, in particular like f_4 in which a rational function may be given, will offer you shortcuts.

→ The first thing you do to a rational function is to split off polynomials by long division, if possible. If not possible, you skip this step. *Here you should pause and think in which of the two cases f_1 and f_2 – either, both, or neither – this step could be carried out. If you don't see it, or don't see how to determine it, ask your neighbor, TA or professor. No, I'm not gonna tell everything here, because I don't want you to do math like a cinema; it wouldn't work out.*

→ The polynomial you have received (if any) can be integrated in a straightforward way, thus accomplishing the easier part of the task. For the remaining term yet to be integrated, the denominator is still as nice or nasty as it was before, but the numerator will have a smaller degree than the denominator now, and this is to be considered as a simplification. *If you don't understand the previous sentence, you have probably dodged your job with the italic sentence in the previous paragraph and deserve no sympathy for it.*

You will now look at the degree of the denominator: *What number is this in the cases of f_1 and f_2 respectively? If you don't know, that's your next question to ask.* The lower it is, the easier the task:

- If the denominator is linear (i.e., of the form $ax+b$), you get a ________. *Think a bit, what kind of function?*[1] As long as you lack the routine of seeing the result at first glance, what you essentially do is to substitute $u = ax+b$.
- If the denominator is quadratic (i.e., of the form ax^2+bx+c with some numbers or parameters a, b, c), you are in for several possibilties. But in any case, you will want to complete squares first: because you know antiderivatives $\int \frac{du}{u^2+a^2}$ and the like, but not $\int \frac{du}{u^2+u}$, so you want to get rid of the linear term in the quadratic polynomial by completing squares.

  You could get arc tan terms from integrating something like $\frac{1}{u^2+a^2}$ or ar tanh or ar coth terms from integrating something like $\frac{1}{u^2-a^2}$. *You'll have to think what I mean by "something like" here, to understand, in what category a given example will fall.* You may also end up with integrating $\frac{1}{u^2}$, and I hope the facts that inverse trigs and hyps have

[1] Answer: Logarithm

been abundant recently hasn't made you forget what you knew already many weeks ago, and that you still can do $\int \frac{du}{u^2}$. And finally, there may be things like $\int \frac{u\,du}{u^2+a^2}$. No this doesn't fit into arc tan or ar tanh, this goes in the ln category: look and see why/how!

You may have to separate fractions, because the separate possibilities in the previous paragraph may come combined. What is $\int \frac{x+1}{x^2+1}\,dx$? When you try to do this example, you'll see what I mean. *If not, ask etc - but you know this strategy already.*

When separating fractions, you don't just separate along where a + sign happens to be. Rather, for instance,

$$\int \frac{-x+4}{(x+1)^2+4}\,dx = \int \frac{-(x+1)}{(x+1)^2+4}\,dx + \int \frac{5}{(x+1)^2+4}\,dx =$$

is the way to go, and you should be able to answer the question, why.

– Ok, now comes the really tough case: If the denominator has a higher degree than two, as, eg., in $\int \frac{dx}{x^3+1}$. How to proceed? *Pause a moment and think about it.* No, don't try too hard, we'll come to this in section 7.3. Then you can fill in the rest here. But let me give away one thing: You have to factor the denominator; and if you can't do it, then most likely, more cannot be done about the integral.

Problems:

Here are a few problems (they are worth many textbook problems in one, so keep your patience with them). You can do them step by step, following the outlined strategy. And once you're through you will have understood the entire strategy. Because they will require you to use all parts of the strategy in their appropriate place, not only single isolated features one at a time.

$$\int_0^1 \frac{x+1}{x^2+x+1}\,dx \qquad \text{and} \qquad \int_0^1 \frac{x^3+x^2+x+1}{x^2-x-1}\,dx$$

It's quite ok if you get stuck somewhere in the middle, like you would also have to stop in an unknown city, pause, assess where you are, and where to go next. All the text above, that's your city map. Study it, follow the directions closely, and see, it should work.

Oh, and here is a question you should now be able to answer: In the strategy, at the place where arc tan and ln occurred (with quadratic denominators), why did I not tell you what to do with a case like $\int \frac{x^2}{x^2+a^2}\,dx$. Did I forget some possibility?

And here we have a less complicated one as a dessert; it will actually prepare you for section 7.3:

Convince yourself that

$$\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\,.$$

The outlined strategy will tell you to write $\int \frac{dx}{x^2-1}$ in terms of inverse *what?*, but $\int \frac{1/2}{x-1} - \frac{1/2}{x+1}$ in terms of logarithms. Carry out the integrals and then compare with what you find at p. 527 of the textbook.

7.1.2: Panorama

You have encountered several techniques exemplified in the textbook, namely substituting, completing squares, separating fractions, reducing improper fractions. Note that they all fit in a particular place of the larger strategy of integrating rational functions. Each of them serves a particular purpose at that particular place, namely some kind of simplification. Have a second look at the strategy and make sure what constitutes the particular simplification achieved in each step. Think of this question as one of the kind "How would I explain the purpose to my dormmate who hasn't understood it yet?" — *Your classmates will come asking you this type of questions, because they are also reading the italic small print, so be prepared. The textbook sucks, but helping out your classmate is fun, even after Valentine's day.*

But lo and behold, the same skills will reappear in strategies for other types of integrals, and they'll serve a similar purpose there as well. *Here is some really funny homework: read your textbook carefully and find all pages which do NOT contain the word 'purpose'. Tear them out and use them to light a cigarette. – No, just kidding, I don't mean to suggest you should start smoking...*

7.1.3: Square roots — Softies may Pose as Tough Guys

As you know how to integrate $\int \sqrt{x}\,dx$ or $\int (1/\sqrt{x})\,dx$, which are just powers, what we need to discuss here is the combination of square roots with other functions. Here, we restrict the discussion to the case where you have to deal with square roots of some more complicated expression. As the scope of our headline is somewhat vague (namely what kinds of integrands are actually covered here, what kinds aren't?), the strategy will also be less organized. But here are a few basic principles:

- If you are dealing with the square root of some "complicated expression", let's call it $u(x)$, try whether you can locate its derivative $u'(x)$ in the integral, as a factor of the integrand. Then you will typically substitute that complicated expression. In such a situation, you may consider the actual difficulty of the problem lower than it should appear at first sight, and the complicated expression is just camouflage: the substitution will get rid of it immediately. Typical situations are like

  $$\int \frac{x}{\sqrt{x^2+1}}\,dx \qquad \text{or} \qquad \int \frac{x^2}{\sqrt{x^3+7}}\,dx \qquad \text{or} \qquad \int \cos x\sqrt{\sin x}\,dx$$

  Yes, do these integrals now. In order to appreciate the method and get the hang of it, compare these with the following examples, which are intentionally set up to *look* similar. But you should view them as *quite different*. In them, the complicated expression under the square root is *not* camouflage, but it's a genuine difficulty:

  $$\int \frac{1}{\sqrt{x^2+1}}\,dx \qquad \text{or} \qquad \int \frac{x^3}{\sqrt{x^3+7}}\,dx \qquad \text{or} \qquad \int \cos x\sqrt{\cos x}\,dx$$

  If you are still looking at them like they are very similar to the above, then look again. You need to have a different notion of "similar" than what you are used to, a notion that doesn't focus on appearance, but on the features relevant for integration. Imagine a new word coined for this concept: "$\int$imilar";-)

For the second set of examples, the first one leads to inverse hyperbolic functions, the other two are something you wouldn't encounter before grad school.

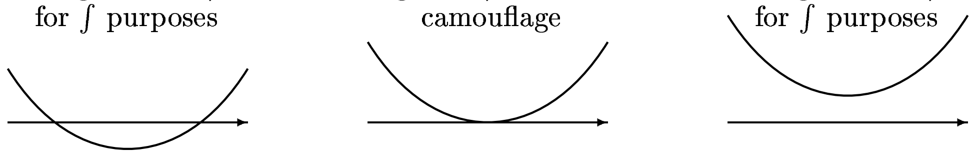
- Ok, the previous item was just how to detect and get rid of the soft guys that try to *look* tough. This skill actually relates to any functions, whose argument is a more complicated expression, like $\int x^2\, e^{x^3+1}\, dx$ or $\int \cos x \ln \sin x\, dx$ and so on. They are the softies. Some tough guys that may *look* similar but are NOT $\int$imilar would be: $\int x\, e^{x^3+1}$ or $\int \sin x\, \ln \sin x\, dx$. Actually the second one isn't that tough either, but it will require a different strategy. *Be sure to see how you would handle $\int \cos x \ln \sin x\, dx$ immediately upon testing against the softies first strategy, whereas you would pass on $\int \sin x \ln \sin x\, dx$ because it fails the softie test at first glance.*

  Note also that $\int x\sqrt{x^2+1}\, dx$ should be considered as a more difficult quiz problem than the weird and messy $\int (2x + \sin x)\sqrt{x^2 - \cos x}\, dx$. *Are you sure you see why? Think of what you would do if you were given either one out of context.*[2]

- Another issue is that the square root itself may be camouflage. Then the text book's "eliminate square roots" skill applies: Like here: $\int \sqrt{x^2}\, dx$, or $\int \sqrt{x^4 + 4x^2 + 4}\, dx$. That's cheap. But the really tricky camouflage is when you have trigs under the square root. You may need trig identities to see that the term under the $\sqrt{\ }$ is actually a square.

  How would you see whether it will work? After all, you won't have the whole mess of trig identities at your breakfast table all the time. And you wouldn't want to try out a bunch of trig' identities at random until, next week, your math professor tells you, hey, these are not the ones to be treated with trig identities. – So it's nice if you can somehow anticipate what will work and what won't before you actually carry out the work. The answer will also demystify the trig identities a bit: You look at the stuff under the square root and try to graph it (assuming it's simple enough so you can do it within the available time): There are three situations:

two (single) zeros: tough under $\sqrt{\ }$ for $\int$ purposes	a "double" zero: good! $\sqrt{\ }$ is just camouflage	no zeros: tough under $\sqrt{\ }$ for $\int$ purposes



  Well, now have a look at the graphs: the middle one looks like $y = x^2$ near the zero, and taking the square root will be almost as easy as taking $\sqrt{x^2}$, even if it's actually a trig' function. — Well, not quite as easy. With trig functions you still have to find the trig identity that does the trick, whereas $\sqrt{x^2}$ is plain. What I really mean, is the square root is as soft (or $\int$oft) as for the case of $\sqrt{x^2}$: you *can* get rid of it. You may have to work for that purpose, which is fair enough, but you'll win. In the tough cases, you couldn't win, in spite of all work.

  So look at the following three cases:

$$\int \sqrt{\cos x + \tfrac{1}{2}}\, dx \qquad \text{or} \qquad \int \sqrt{\cos x + 1}\, dx \qquad \text{or} \qquad \int \sqrt{\cos x + 2}\, dx$$

[2] Answer: They are equally soft, but the first one's camouflage is much better

For the middle one, you look for a trig identity (indeed, $\cos x + 1 = 2\cos^2(\frac{x}{2})$), but the other two you leave alone (or leave them to some whiz at grad school).

And remember that

$$\int \sin x\sqrt{\cos x + \tfrac{1}{2}}\,dx \qquad \text{or} \qquad \int \sin x\sqrt{\cos x + 2}\,dx$$

are softies of the previous category (ie., nothing *but* a square root, after substitution), whereas

$$\int \cos x\sqrt{\cos x + \tfrac{1}{2}}\,dx \qquad \text{or} \qquad \int \cos x\sqrt{\cos x + 2}\,dx$$

are tough and you don't touch them. And with

$$\int \sin x\sqrt{\cos x + 1}\,dx \qquad \text{or} \qquad \int \cos x\sqrt{\cos x + 1}\,dx$$

the square root itself is camouflage: so you get rid of it first, and even if a non-obvious integral still remains, you'll call this first step progress already, and then you can still see and think what would be appropriate for further treatment of that integral.

Now, have a second look at the pictures: Do you understand, why you don't have a chance to write $\cos x + \frac{1}{2}$ *as a complete square? If so, explain it to your classmate who hasn't understood yet. If not, ask your classmate who has.*

Have yet another look at the pictures: Do you understand, why you don't have a chance to write $\cos x + 2$ *as a complete square either? – Probably no. You are* <u>*not*</u> *expected to, and it's not obvious at all. But it's good to know that the trick with the zeros works nevertheless. And I did expect you to come up with that question at that point.*

Now, copy all examples on separate sheets or filecards, in random order, put them away for a day at least, then do all those integrals that you can do, to the extent that you can be expected to do them, and recognize those which you could not be expected to do as untractable.

7.1.4: Genuine Square Roots – No Camouflaged Softies

Are you sure you have checked for the two types of camouflaged soft cases? In the text book the corresponding skills are called 'try a substitution' and 'eliminating a square root'. Keep remembering: What may look similar, need not be $\int$imilar!

A lot of stuff can be under a square root, most of which will make the integrand rather resistant again simplification attempts. Here we deal only with polynomials and trigs or hyperbolics:

– With polynomials under the square root, you look for the degree of the polynomial:

  <u>Linear</u> functions yield roots (powers with fraction exponent) again:
  $\int \sqrt{2x+1}\,dx = \frac{1}{3}(2x+1)^{3/2} + C$, or $\int (1/\sqrt{2x+1})\,dx = \sqrt{2x+1} + C$.

  <u>Quadratic polynomials</u>: guess what!? You complete the squares again! *Guess why? – Same reason as in the case of rationals; Look it up, if you don't remember it.* For the time being, let me assume the simplest situation, where a root of a quadratic polynomial is

in the *denominator*, and the *numerator is constant*. And then, it will look like either of $\int du/\sqrt{u^2+a^2}$ or $\int du/\sqrt{u^2-a^2}$ or $\int du/\sqrt{a^2-u^2}$. And you get inverse trigs or inverse hyps, namely ar sinh or ar cosh or arc sin respectively.

However, if you have something like $\int x/\sqrt{x^2+4}\,dx$, the essential feature being that you have an x in the numerator, then... — Hey, you shouldn't have this at this place any more! Reread the first paragraph of this section!

But watch out, genuine tough guys may hide next to softies that are only camouflaging as tough guys...
For instance, in

$$\int \frac{x+1}{\sqrt{x^2+1}}\,dx$$

you have to do *what??* to separate the softies from the tough guys? *No, I won't tell you, you think a bit and, if unsuccessful, ask your neighbor or TA or professor etc*

Actually, if the square roots just mentioned occur elsewhere, say, in the numerator, you are still in for the same type of inverse trig or hyp functions, but you cannot read the antiderivative off immediately. See the section on integration by parts or the one on trig' substitutions for more. But then again, don't be worried: You are not required to handle anything that could be done. So, if you see

$$\int \frac{\sqrt{x^2+5x+10}}{x+5}\,dx$$

and know that inverse hyperbolic functions should apply, but still couldn't get through to the end, you should call this knowledge genuine progress already, and you will learn the next step later (in 7.4 for instance).

Polynomials of higher degree under the square root are left alone, unless...

What could you do to simplify

$$\int \sqrt{x^3-3x+2}\,dx\ ?$$

Look at the figure in 7.1.3, and take, as a hint, that in contrast, you wouldn't touch $\int \sqrt{x^3-3x+3}\,dx$ at all.

– Integrals containing square roots of trig expressions won't simplify, unless they actually fit in the camouflage types of 7.1.3.

Now here is a question worth thinking of: You know, when you see $\sqrt{\textit{some quadratic}}$ *under an integral, then, by competing squares, you can transform it into one of the forms* $\sqrt{u^2+a^2}$, $\sqrt{u^2-a^2}$, *or* $\sqrt{a^2-u^2}$. *And you know that these will lead to the inverse trig and inverse hyp functions* arc sin, ar sinh, and ar cosh, *but of course you find it absolutely disgusting to memorize which to which. So you draw very rough sketches of all of these functions, in particular looking at the relevant domains of definition, and then... — uhm, wait a minute — hey, that cannot match any other way, but...*

7.1.5: Trig Functions

Trig functions are always a mess, because they come under different names that hide their relations. If you have sec, tan, cot, sin, cos all mixed up, you won't see any structure in any expression any more. Either they have prepared the examples with all these functions to make it easy on you (because they expect you to recognize $\sec^2$ as the derivative of tan), or to annoy you, because they want to see whether you can find your way through the mirages of trig functions. In the latter case, here is a strategy that often works: Get rid of sec, csc, cot first (writing them as $1/\cos$, $1/\sin$, $1/\tan$). If you can write everything in terms of tan, fine. Else get rid of tan as well, replacing it by $\sin/\cos$.

All this has nothing to do with integrals. The special tricks in the textbook are just this: special tricks. And don't let yourself being talked into the story that the textbook author invented the method of writing 1 in the form $\frac{\sec x+\tan x}{\sec x+\tan x}$ by sheer ingenuity to solve $\int \sec x\,dx$. Once upon a time somebody found out the antiderivative of $1/\cos$, I don't know who. Probably one of the forefathers and founders of calculus in the 17th century. Since then, we math folks just know it can be done, and we know the result (or know where to look it up). And you bet, it can be written in many different ways, like eg. $\ln(\cos\frac{x}{2}+\sin\frac{x}{2})-\ln(\cos\frac{x}{2}-\sin\frac{x}{2})$, or $\ln\tan(\frac{x}{2}+\frac{\pi}{4})$, or $\ln(\sec x+\tan x)$.[3] Once you have the result in mind, you can invent the method to set up the magician's performance. And bingo, that's your textbook example. No, that's not fair on you. How should you ever have invented sumething as utterly absord as to write 1 as $\frac{\sec x+\tan x}{\sec x+\tan x}$? Neither did the textbook author. Ok, tit for tat, and this is why I have given away the plot: they worked their way backwards from a previously known result. This is why I have given away the plot. To give away the plot is not fair on the magician either. So, now you are even with him.

[3] I am assuming $x \in (-\pi/2, \pi/2)$ here, so I needn't bother about the absolute value signs.