— Retrospective on some features concerning series
— Jochen Denzler, Apr 2001

Picard's Method: Why? How? How Not?

This has little to do with series, but some confusion seems to have arisen from this item; an item which was merely intended as an example for recursively defined sequences. But it is a worthwhile example nevertheless.

You learnt Newton's method as a tool for solving equations $f(x) = 0$ numerically. Just to remind you: If you start anywhere, say, call the start point x_0, and then define the sequence $\{x_n\}$ recursively by $x_{n+1} = x_n - f(x_n)/f'(x_n)$, then the sequence $\{x_n\}$ "should" converge to a solution of the equation $f(x) = 0$. "should" means that (unless $f(x) = 0$ has no solutions) it will converge except for some very rare exceptional starting points x_0. If this happens, just try a different starting point, and the odds are it will converge then. Moreover, it will converge very rapidly (unless by bad luck $f'(x_*) = 0$ as well for the solution x_* it converges to, in which case the convergence will be slowed down). And if your start point is already close enough to such a solution, convergence is actually assured, no exceptional points there.

Now you have learnt Picard's method as a tool for solving equations $x = g(x)$ numerically: You start anywhere, call the start point x_0, and then define the sequence $\{x_n\}$ recursively by $x_{n+1} = g(x_n)$. Clearly, you will be out of luck again, if $x = g(x)$ has no solution, but on top of this, you are sure to be out of luck, if for every solution x_* to $x = g(x)$, it holds $|g'(x_*)| > 1$. (We haven't explained why this is the case, the book gives a hint in ex. 9 on p. 625, but it's a fact). When Picard's method converges, it will converge to a solution, but typically much more slowly than Newton's method.

So, why the heck, you may ask, would anybody want to use Picard's method when Newton's method is superior in every respect?

Bad answer: *Because they handle different types of equations.* — No. If you have, for instance, the equation $x = \cos x =: g(x)$, you rewrite it in the blink of an eye as $f(x) := x - \cos x = 0$, and Newton's method will be more than happy to deal with it. And if you compare how many steps, starting from $x_0 = 0$, it takes to get 10 digits precision, you'll see the superiority of Newton's method.

Bad answer: *It's probably only of historic interest; they used Picard, until Newton found something better.* — No, assuming they are named after the right people, Newton lived much earlier. After all (no, before all!), Newton is one of the creators of Calculus, who would antedate him? I couldn't promise that in some stray context, some Greek wouldn't have done something that ressembles Picard's method, but systematically, Newton's method was available as soon as Calculus was available, nobody had to wait patiently for it to come and grind their teeth at Picard's slow convergence in the meanwhile.

Better guess, but still wrong: *ok, the publisher needs to fill 1200 pages, so they dig out poor Mr Picard's dead body and insert a zombie chapter. Happens all the time* — Good understanding of textbooks, but not so good an understanding of Mathematics. Not your fault, you couldn't be expected to know this one.

A simple good answer: *Oh, it's so simple conceptually, it's just interesting we can often do it without derivative. It's not practical, but it's pretty.* — Good understanding of mathematical aesthetics, but we usually take a more practical approach here, in spite of ourselves. (Wonder, why calculus professors don't suffer from gastritic troubles more often. But they do get gray hair in early years sometimes, from the grief of their sense of beauty being suppressed.)

A very good, unexpected, answer: *Picard's method is a giant disguising as a dwarf.* — When the really tough problems come, then Picard will have its big time. In really tough problems, the division in Newton's method (you have to calculate $f(x_n)/f'(x_n)$) is hard. No kidding, I really mean you may encounter situations where dividing is harder than finding a deriviative. But not in one-variable calculus. In multivariable calculus maybe.

In one-variable calculus, evaluating $q = 7/9$ means solving the equation $9q = 7$. Peanuts. In two-variable calculus (to start next semester), what corresponds to division, would be to solve two equations in two variables, like $9q + 3r = 7, 4q - 7r = 3$. Not peanuts, but still hazelnuts. With 100 variables: well, we have the computer as a nutcracker, so what. But the 20th century has witnessed the rise of infinitely-many variable calculus (not under this name), and it is a machinery that deals with solving differential equations (among other stuff), including the tough ones we omitted. And this is where Picard's method becomes the tool of choice, not even mainly for numerical purpose, but already for a theoretical understanding. You may get a glimpse of its use in later undergrad maths, depending on details of your curriculum; or maybe not.

So here we have this power tool. And the fact that it's so easy to press the cosine key on a pocket calculator over and over again, makes it a very good example. Hey, we collect the examples for you not from the dustbin of history, but from the royal trasure! Join our pride!

Convergence tests for series

This is meant as a retrospective. I am assuming you have already gone through the material covered here, and I am only collecting and organizing what you (should) have learnt. This will be no substitute for the textbook.

Focus

In the context of series, we put some stress on determining whether a series converges or not, but very little stress on actually finding the sum. The reason is that finding the actual sum can typically be done only on a case to case basis, if at all: so there are no general techniques to be learnt. In many cases, it is simply not possible to come up with a neat expression for the actual sum of the series. On the other hand, many series are quite useful tools for their own sake, so that you can use them unevaluated, as soon as you know they make sense (ie., converge). [Looking ahead for a moment: This latter statement applies mainly to series of functions, in particular power series, less so to series of numbers.] With this in mind, let us sort systematically the whole bunch of convergence tests you have learnt or are about to learn:

Essentially there are only three tests, two small ones and a big one. The big one comes as a crowd with many special pre-cooked versions.

One, and the Crowd, and Another One

→ The first (small) test is actually a *divergence* test: it says $\sum a_n$ cannot converge, unless $a_n \to 0$. It merely screens out those series that never had a chance to converge. It wouldn't screen out the harmonic series $\sum \frac{1}{n}$ (because indeed $\frac{1}{n} \to 0$ as $n \to \infty$), but still the harmonic series is divergent. This test would never positively declare a series as convergent.

It is kind of true that a series $\sum a_n$ *does* converge, if $a_n \to 0$ *fast enough* as $n \to \infty$. But this becomes really true only if you make precise what you mean by "fast enough". But this is essentially our second test already.

→ The second (the big) test is the comparison test. In order to use it, you need a certain collection of particular series known to converge or diverge. It is with them that a given series can be compared. These "yardstick" series are usually $\sum_n 1/n^p$ (convergent for $p > 1$, divergent otherwise) and the geometric series $\sum_n ar^n$ (convergent for $|r| < 1$, divergent otherwise). In certain subtle cases you may need other "yardstick" series in addition to these, but when you actually come to a situation where you need these, you will probably have enough mathematical maturity already to come up with these extra series. I will detail out the comparison tests below.

Originally, the comparison tests work only for series of nonnegative terms. However, if you are given a series $\sum a_n$ containing negative terms as well, you will first examine the series $\sum |a_n|$ instead. You have learnt in class that if $\sum |a_n|$ converges (as would typically be shown using any of the comparison test variants), then so does $\sum a_n$. And that actually in this case $\sum a_n$ would be called *absolutely* convergent, which is an interesting property of its own and is better than convergence.

→ Finally, there is Leibniz' test for alternating series: If $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$ and $a_n \to 0$, then $a_1 - a_2 + a_3 - a_4 + - \ldots$ converges. Leibniz' test can detect convergence in this situation, even if absolute convergence doesn't hold. If an alternating series happens to be absolutely convergent as well, Leibniz' test wouldn't detect it.

The name-maimers, a secret and sinister group of unknown individuals that has eluded both the Calculus Intelligence Agency and the Federal Bureau of Orthography started a conspiracy to misspell the name of Leibniz. If you ever encounter a trace of this misconduct of theirs, don't trust them. The textbook is right this time. No 't' in Leibniz.

A Closer Look at the Comparison Test Crowd

The integral test consists of comparing a sum with an improper integral. Using this test establishes the p-series $\sum_n 1/n^p$ and possibly the geometric series $\sum_n ar^n$ as "yardstick series"[1] for comparison with other series. As an alternative way to decide on convergence vs. divergence of the geometric series, we can go back to the definition, because there is an explicit formula for the partial sums of geometric series. *If this sounds new to you, go and find this formula in your notes or the textbook.*

Compare!

The basic comparison test says: If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then so does $\sum a_n$. If $0 \leq c_n \leq a_n$ and $\sum c_n$ diverges, then so does $\sum a_n$. You use this with a_n given (complicated) and have to decide first, whether you believe that $\sum a_n$ converges (and want to establish it

[1] yardstick series is NOT an established mathematical term. I am coining it ad hoc to describe the purpose of such series.

now) or whether you believe that it diverges and want to establish that now. If you don't know what to believe, you have to try both. So you have to find either b_n od c_n (or try both) with simpler formulas so that you can recognize $\sum b_n$ or $\sum c_n$ as a "yardstick series" satisfying the comparison conditions.

Handling inequalities (like eg., showing $\frac{n+1}{n^2+1} \geq \frac{1}{n}$ for all n) is a task you will rarely have trained, and so, except for really obvious cases, you will find even the easier examples somewhat difficult. They wouldn't involve sophisticated manipulations, but unless you have a good feeling for the territory already, you wouldn't know where to put the next step. When you see inequalities used, you will usually find it easy to assess the legitimacy of each single calculational step. The difficulty will be to see the motivation. Try to cope with this difficulty in every single example you encounter. It is the mathematical version of the wisdom that can tell the important from the unimportant, the wisdom that recognizes a red herring if there is one. Ponder about the "Why does he...?" and "Why not...?" questions whenever they arise.

While you find the direct comparison difficult, the limit comparison method comes in handy. It's less powerful though, as far as applicability is concerned: Whenever limit comparison applies, direct comparison could also be used (maybe much less conveniently), but not vice versa.

If you are an A or upper B student and can spend some time, you may want to look for a direct comparison argument whenever they suggest you the (often simpler) limit comparison argument. Otherwise, you may want to try the same not always, but occasionally, in some simpler-looking cases. Because equations are dull; whereas inequalities breed the wisdom to distinguish.

Root and Ratio:
The ratio and root tests are ready-to-use versions of comparison with the geometric series. So the yardstick is already built in. In those cases where the geometric series is too coarse a yardstick, ratio and root tests will be inconclusive, and you would go back to the (direct or limit) comparison test with a comparison series different from a geometric one. In particular if a_n is a rational function of n, do NOT use the ratio test (even if this sounds like paradox). It would automatically come out inconclusive. (And so would the root test.) Use root or ratio test, if a_n contains n in an exponent or contains factorials involving n.

Convergence is determined by how fast a_n goes to 0. $1/n$ isn't fast enough, $1/n^2$ is. The borderline is somewhere in between, very close to $1/n$. And this refers to the behavior for *large* n; we don't care for the first few zillion terms. The terms in a convergent geometric series go to 0 VERY rapidly (they may have a slow start). If you want to experience this speed, punch $1/n^4$ and $(0.9)^n$ in your pocket calculator, for $n = 10, 30, 100, 300, 1000, 3000$ and compare the results. Trying the ratio test on $\sum 1/n^4$ would be like measuring the speed of a snail with a highway patrol speed radar![2]

The root test is essentially the same as the ratio test, as far as the ingredients are concerned. When it comes to using it, the root test is more powerful in applicability: Whenever the ratio test works, the root test will (in principle) also work, but not vice versa. However, in many cases you may prefer the ratio test, because the practical calculations will be a bit easier. If you go for the root test and a_n contains factorials, you should know that $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$. Why don't you just add this result to Table 8.1 on p. 616 of the text book and merge it with your

[2] Actually, this comparison is not quite fair: because it's an understatement. Well, not really an understatement; because calling it an understatement would be an understatement. Or rather, it would be, what couldn't be called an understatement, unless by understatement. And so on, you get the idea... Geometric series, and $\sum n^{-p}$ are really yardsticks from different worlds

other favorite formulas! — I owe you an explanation why $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$. Well:

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \geq 1 \cdot 1 \cdot 1 \cdot 1 \cdot 4 \cdot 4 \cdot 4 \cdot 4$$

and similarly in general $n! \geq (\frac{n}{2})^{n/2}$. (This is another instance of inequality trouble: The calculation is probably obvious, but to see why you would/should do it requires knowing the territory and anticipating the end.) Therefore, $\sqrt[n]{n!} \geq \sqrt[n]{(n/2)^{n/2}} = (n/2)^{1/2} \to \infty$.

An Erratum

On page 607, the text book claims: "Laplace used infinite series to prove the stability of the solar system." Unless the textbook author(s) have dug out some hitherto undiscovered manuscript by Laplace, this claim is wrong. It might be true that Laplace hoped to achieve such a result, I wouldn't know. He certainly was in this type of questions. But it was only later Poincaré who made clear at least why the problem is actually a very tough one. Roughly about 1900 (I am too lazy too look up the exact year), he got a prize for understanding the question at least! The answer is still not fully known! I do not know who has spread the rumour about Laplace's proving the stability of the solar system. Somebody might guess the statement is due to a transmission error on a poor phone line, and the actual statement should have read "Laplace used infinite weariness to prove the imbecility of the phony teaching." But don't quote me with this, it's just a very wild guess.

If you want to learn about the persistence of hoaxes in science in a competely different (non-math) context, you may want to find the book "The great Eskimo vocabulary hoax, and other irreverent essays on the study of language" by Geoffrey Pullum in some library (ND doesn't hold it) and read the half-dozen pages of the chapter that gave the book the title. (I can dig out a copy for you, if you like and give me some time.) If you know other stuff I could have quoted here, let me know. Happens all the time. To err is human, but let's fix errors, when detected; no excuse.