— Third manuscript on integrals, expanding on 7.3 of our textbook; more to follow — Jochen Denzler Feb 2001

## 7.3: Partial Fractions

Partial fraction decomposition is a method of algebraically transforming rational functions into a certain standard form that makes integration a matter of routine. For instance, we would transform

$$\frac{x^3+x-3}{x^4+x^3-x^2+x-2} = \frac{x^3+x-3}{(x-1)(x+2)(x^2+1)} = \frac{-1/6}{x-1} + \frac{13/15}{x+2} + \frac{\frac{3}{10}(x+3)}{x^2+1} \tag{1}$$

The main task and main work consists of understanding why we write it this way, and how we can find this form. Once you have written the integrand in this way, integration reduces to doing standard integrals you have seen already. Namely,

$$\int \left( \frac{-1/6}{x-1} + \frac{13/15}{x+2} + \frac{\frac{3}{10}(x+3)}{x^2+1} \right) dx = -\frac{1}{6} \ln|x-1| + \frac{13}{15} \ln|x+2| + \frac{3}{10} \left( 3 \arctan x + \frac{1}{2} \ln(1+x^2) \right) + C$$

Compare with 7.1: There, I introduced a procedure to integrate rational functions, but only with denominator polynomials of degree one and two. Our approach here will enable us to handle higer degree denominators as well, and at the same time, gives a slight variant of the case with quadratic polynomials.

→ As in 7.1., you will only handle *proper* fractions; improper fractions, i.e., rational functions whose numerator has a degree higher or equal to the denominator will be reduced by long division, splitting off a polynomial that can readily be integrated immediately.

→ Now you have to factor the denominator. This follows the general
**Theorem: Any polynomial can be written as a product of linear and quadratic polynomials.**
This may be difficult to carry out in practice, and if you get stuck here, you can't do much about the integral either. You will have to apply this theorem to the denominator of the rational function. To do so was the first step in $\int$implification (1) of our example.

Let us pause a bit here: If you want to write a quadratic polynomial, e.g., $x^2+4x+3$ as a product of linear polynomials, you can always use the quadratic formula to find the zeros of that polynomial: $x^2+4x+3=0$ if and only if $x=-3$ or $x=-1$. This is how you find $x^2+4x+3=(x+1)(x+3)$. If the quadratic formula does not give any real zeros, as in the case of $x^2+4x+5$, you leave the quadratic polynomial alone. Zeros of the polynomial will always correspond to linear factors.

In the case (1), you have no feasible systematic way to find zeros of the denominator $x^4+x^3-x^2+x-2$. By guessing, you may however find that $x=1$ is a zero, and then you know

$$\begin{aligned} x^4+x^3-x^2+x-2 &= (x-1)(\text{poly' of deg 3, to be found by long division}) \\ &= (x-1)(x^3+2x^2+x+2) \end{aligned}$$

If you can guess another zero of the remaining factor $(x^3 + 2x^2 + x + 2)$ –here, this would be $x = -2$–, you get $(x^3 + 2x^2 + x + 2) = (x + 2)(x^2 + 1)$

In order to see how surprisingly strong this factorization theorem is, try the polynomial $x^4 + 1$. It has no zeros, so applying our theorem to it cannot produce linear factors. So, if our boldfaced theorem is true, it must be possible to write $x^4+1$ as a product of two quadratic polynomials. If you try to find how this will actually look: well, it will be quite sophisticated, you would probably not guess it. You have to find numbers $p_1, q_1, p_2, q_2$ such that $x^4+1 = (x^2+p_1x+q_1)(x^2+p_2x+q_2)$. *Can you do this, by expanding the right hand side and comparing coefficients of powers of $x$? — In principle you can; but don't get disheartened: it takes some time to find the coefficients.* If you have actually carried it out, you'll be in for a surprise:[1]

We can now continue our itemized strategy of finding a partial fraction decomposition:

→ **Theorem: Any proper fraction of polynomials can be decomposed into partial fractions according to the following example, which displays all features that could occur:** Given numbers $a_1, a_2, p_1, p_2, q_1, q_2$, and any polynomial in the numerator, numbers $b_1, b_2, b_3 \ldots$ can be found such that:

$$\frac{\text{any polyn' of degree less than the degree of the denominator}}{(x-a_1)(x-a_2)^4(x^2+p_1x+q_1)(x^2+p_2x+q_2)^2} =$$

$$= \frac{b_1}{(x-a_1)} + \frac{b_2}{(x-a_2)} + \frac{b_3}{(x-a_2)^2} + \frac{b_4}{(x-a_2)^3} + \frac{b_5}{(x-a_2)^4} +$$

$$+ \frac{b_6x+b_7}{x^2+p_1x+q_1} + \frac{b_8x+b_9}{x^2+p_2x+q_2} + \frac{b_{10}x+b_{11}}{(x^2+p_2x+q_2)^2}$$

In other words:

- For every nonrepeated linear factor in the denominator on the left (here $x - a_1$), you get one simple fraction (here, number$/(x - a_1)$) on the right.
- For every repeated linear factor in the denominator on the left (here $(x - a_2)^4$), you get as many simple fractions on the right as the number how often that factor was repeated, and their denominators echo the corresponding factor from the left, but with increasing powers from 1 up to what we had on the left.
- For every nonrepeated quadratic factor in the denominator on the left (here $x^2 + p_1x + q_1$), you get one simple fraction with that very denominator on the right. The denominator of that fraction may be a linear polynomial now: $(b_6x + b_7)/(x^2 + p_1x + q_1)$.
- For every repeated quadratic factor, you get similar fractions on the right, with increasing powers in the denominator.

What remains to be done is to see how yo can actually find $b_1, b_2, \ldots$. This will be described in a moment. Once you have accomplished this, you can do the integral term by term. Actually, the standard integrals available to you cannot handle the last term yet, and you may be content with the assurance, for the time being, that you will not encounter cases leading to this kind of term.

[1] Answer: $x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$. At least, check this by expanding.

This ends the basic outline of partial fraction decomposition. We now discuss the way how you can actually find the numbers $b_1, b_2, b_3, \ldots$.

There is a simple-minded way that always works; however, in all but the simplest cases it will be rather tedious. But you should have understood it and tried for yourself, before you venture into the more sophisticated, but very fast way of doing it. For example, we need to find $b_1, b_2, b_3$ such that

$$\frac{x^2+2x+2}{x(x-1)(x+2)} = \frac{b_1}{x} + \frac{b_2}{x-1} + \frac{b_3}{x+2} \tag{2}$$

holds identically (for all $x$). So we bring the right hand side on a common denominator, and sort powers of $x$ in the numerator:

$$\begin{aligned}
\frac{b_1}{x} + \frac{b_2}{x-1} + \frac{b_3}{x+2} &= \frac{b_1(x-1)(x+2)}{x(x-1)(x+2)} + \frac{b_2 x(x+2)}{x(x-1)(x+2)} + \frac{b_3 x(x-1)}{x(x-1)(x+2)} \\
&= \frac{b_1(x^2+x-2) + b_2(x^2+2x) + b_3(x^2-x)}{x(x-1)(x+2)} \\
&= \frac{x^2(b_1+b_2+b_3) + x(b_1+2b_2-b_3) + (-2b_1)}{x(x-1)(x+2)} \\
\text{and this should } &= \frac{x^2+2x+2}{x(x-1)(x+2)}
\end{aligned}$$

So comparing coefficients in the numerator, you need

$$\begin{array}{rcl}
b_1 + b_2 + b_3 = 1 & \searrow & \\
 & \oplus \longrightarrow & 2b_1 + 3b_2 = 3 \\
b_1 + 2b_2 - b_3 = 2 & \nearrow & \\
-2b_1 \qquad\quad = 2 & \longrightarrow & b_1 \qquad = -1
\end{array}$$

Therefore we get $b_1 = -1$, $b_2 = 5/3$, $b_3 = 1/3$.

This method is available in all cases, but it involves as many equations in as many unknowns as is specified by the degree of the denominator. — In contrast, here is a shorter method, which will however only apply to linear nonrepeated factors. With a slight modification for linear repeated factors, it will only give the coefficient of the highest power (in the big example given above, it would therefore only yield $b_1$ and $b_5$). For back reference in the future: The method could be generalized to quadratic factors, using intermediate calculations with complex numbers (for those of you who happen to know complex numbers from school). However, I will not detail this out, as it would lead too far away from the focus of our freshman calculus course. So, I will now assume we have nonrepeated linear factors only:

To determine $b_1$, $b_2$, $b_3$ *in turn* from (2), we multiply (2) by the corresponding denominators respectively, namely by $x$, $x-1$ and $x+2$. This is done in separate independent steps starting over from (2) each time. For $b_1$, multiplication by $x$ transforms (2) into

$$\frac{x^2+2x+2}{(x-1)(x+2)} = b_1 + \left(\frac{b_2}{x-1} + \frac{b_3}{x+2}\right) \cdot x$$

We (pretend to) plug in $x = 0$ into this equation, and get

$$\frac{0^2 + 2 \cdot 0 + 2}{(0-1)(0+2)} = b_1 + (\ldots) \cdot 0$$

i.e., $-1 = b_1$ immediately. The method is called cover-up method, because it can be done without a lot of writing already from (2): To obtain $b_1$, you look at the left hand side, cover up exactly that term in the denominator that goes with $b_1$ on the right hand side, and then you plug in that number for $x$ which would have made vanish the covered-up factor in the denominator.

I have been careful to say we *pretend to* plug in, rather than 'we plug in'. The reason is that $x = 0$ is not legitimate to plug in into (2), exactly because of the vanishing denominator. Therfore, an equation derived from (2) is also not legitimate to be used for $x = 0$. What we actually mean to do here is to calculate the *limit* as $x \to 0$. But the actual calculation of this limit will now (i.e., after having multiplied by $x$) amount practically to plugging in $x = 0$.

There is kind of a philosophical message coming together with the partial fraction decomposition. You should consider those points $x$ of a rational function $f$ as its distinctive marks, where the denominator vanishes. If ever rational functions were wanted by the sheriff for wrongdoing, their vertical asymptotes would be the information given on the public announcement :-) With some embellishments added, there will be a result in advanced calculus to the effect that the behavior of a rational function near these points identifies that function nearly as uniquely as a fingerprint. To write a rational function in terms of partial fractions means to write it in such a way as to display certain of its essential features the most visibly. Displaying essential features as clearly as possible will simplify any scrutiny, in particular the search for an antiderivative. And the cover up method is so smart and efficient just because it uses those numbers for $x$ where the essential things happen, namely where the denominator of the rational function vanishes. By focusing on the essential points (in the example $x = 0$, $x = 1$ and $x = -2$) – 'essential points' in the literal as well as in the figurative sense – we avoid unnecessary calculations and retreive $b_1$, $b_2$ and $b_3$ exactly at those places where they naturally belong.

Your textbook will give you a few more tricks how to calculate coefficients in cases where repeated factors occur.

## Some practice problems

Here are a few examples for practising:

$(a)$ $$\int_0^1 \frac{dx}{x^3+1}$$

$(b)$ $$\int_2^4 \frac{x^2+2x+3}{(x-1)^2(x^2+1)}\,dx$$

$(c)$ $$\int_0^1 \frac{x^4+1}{x^2+1}\,dx$$

$(d)$ $$\int_0^1 \frac{dx}{x^4+1}$$ this is a tough one, because of the difficulty of factoring

$(e)$ $$\int_0^1 \frac{x\,dx}{x^4+1}$$ this one is much easier – simplify the integral by a substitution first!

$(f)$ $$\int_0^1 \frac{x+1}{x^2+x+1}\,dx$$

## Solution to 7.2.5

The equation

$$\int e^x \cosh x\,dx = e^x \cosh x - \int e^x \sinh x\,dx = e^x \cosh x - e^x \sinh x + \int e^x \cosh x\,dx \qquad (*)$$

is an equation of *indefinite* integrals. These do not represent a single function $F(x)$, but a whole family of antiderivatives, $F(x)+C$. Namely, $\int e^x \cosh x\,dx = \int \frac{1}{2}(e^{2x}+1)\,dx = \frac{1}{4}e^{2x} + \frac{1}{2}x + C$. If you plug this in on both sides of $(*)$, you have to be aware that the constant $C$ may be different in the two occurrences of the indefinite integral. After cancelling this integral, what we are really left with is *not* $0 = 1$, but rather $0 = 1 + C$ for some constant $C$, and there is nothing wrong with this result.

## Solution to practice problems

Not all intermediate steps are carried out, but essential steps are given.

(a)

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2-x+1}$$

with $a = \frac{1}{3}$ (cover up method), $b = -\frac{1}{3}$, $c = \frac{2}{3}$ (solve linear equations for unknown coefficients). Must complete square in denominator and separate fractions in order to reduce to standard integrals:

$$\frac{1}{3}\left(\frac{-x+2}{x^2-x+1}\right) = \frac{1}{3}\left(\frac{-x+2}{(x-\frac{1}{2})^2+\frac{3}{4}}\right) = \frac{1}{3}\left(\frac{-(x-\frac{1}{2})}{(x-\frac{1}{2})^2+\frac{3}{4}} + \frac{\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}\right)$$

$$\begin{aligned}\int_0^1 \frac{dx}{x^3+1} &= \frac{1}{3}\int_0^1 \left(\frac{1}{x+1} - \frac{(x-\frac{1}{2})}{(x-\frac{1}{2})^2+\frac{3}{4}} + \frac{\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}\right) dx = \\ &= \frac{1}{3}\left([\ln(x+1)]_0^1 - \left[\frac{1}{2}\ln\left((x-\frac{1}{2})^2+\frac{3}{4}\right)\right]_0^1 + \left[\frac{3}{2}\frac{2}{\sqrt{3}}\arctan\frac{x-\frac{1}{2}}{\sqrt{3}/2}\right]_0^1\right) \\ &= \frac{1}{3}\left(\ln 2 + 0 + \sqrt{3}\arctan(\frac{1}{\sqrt{3}}) - \arctan(-\frac{1}{\sqrt{3}})\right) = \frac{1}{3}\left(\ln 2 + \frac{\pi}{3}\sqrt{3}\right)\end{aligned}$$

(b)

$$\begin{aligned}\int_2^4 \frac{x^2+2x+3}{(x-1)^2(x^2+1)}\,dx &= \int_2^4 \left(\frac{3}{(x-1)^2} - \frac{1}{(x-1)} + \frac{x-1}{x^2+1}\right) dx = \\ &= \left[\frac{-3}{(x-1)} - \ln|x-1| + \frac{1}{2}\ln(x^2+1) - \arctan x\right]_2^4 = 2 + \arctan 2 - \arctan 4 + \frac{1}{2}\ln\frac{17}{45}\end{aligned}$$

(c)

$$\int_0^1 \frac{x^4+1}{x^2+1}\,dx = \int_0^1 \left(x^2 - 1 + \frac{2}{x^2+1}\right) dx = \frac{1}{3} - 1 + 2\arctan 1 = \frac{\pi}{2} - \frac{2}{3}$$

(d)

Factorization of denominator: see page 2.

$$\frac{1}{x^4+1} = \frac{ax+b}{x^2+\sqrt{2}x+1} + \frac{cx+d}{x^2-\sqrt{2}x+1}$$

With the tools available here, there is probably no shortcut to solving the equations for $a, b, c, d$:

$$\begin{array}{ll} x^0: & b+d=1 \\ x^1: & a+c+(d-b)\sqrt{2}=0 \\ x^2: & b+d+(c-a)\sqrt{2}=0 \\ x^3: & a+c=0 \end{array} \qquad \begin{array}{l} \text{“}x^3\text{”}\&\text{“}x^1\text{”}: \quad d=b \\ \Longrightarrow \text{(with “}x^0\text{”)} \quad b=d=\tfrac{1}{2} \\ \Longrightarrow a=\sqrt{2}/4=-c \end{array}$$

$$\int_0^1 \frac{(\sqrt{2}/4)x+1/2}{x^2+\sqrt{2}x+1}\,dx = \frac{\sqrt{2}}{4}\int_0^1 \left(\frac{x+\sqrt{2}/2}{(x+\sqrt{2}/2)^2+1/2} + \frac{\sqrt{2}/2}{(x+\sqrt{2}/2)^2+1/2}\right)dx =$$

$$= \frac{\sqrt{2}}{8}\left[\ln\Big((x+\sqrt{2}/2)^2+1/2\Big)\right]_0^1 + \frac{\sqrt{2}}{4}\left[\arctan\frac{x+\sqrt{2}/2}{\sqrt{2}/2}\right]_0^1 =$$

$$= \frac{\sqrt{2}}{8}\ln(2+\sqrt{2}) + \frac{\sqrt{2}}{4}\Big(\arctan(1+\sqrt{2}) - \frac{\pi}{4}\Big)$$

Similarly,

$$\int_0^1 \frac{(-\sqrt{2}/4)x+1/2}{x^2-\sqrt{2}x+1}\,dx = -\frac{\sqrt{2}}{4}\int_0^1 \left(\frac{x-\sqrt{2}/2}{(x-\sqrt{2}/2)^2+1/2} - \frac{\sqrt{2}/2}{(x-\sqrt{2}/2)^2+1/2}\right)dx =$$

$$= -\frac{\sqrt{2}}{8}\left[\ln\Big((x-\sqrt{2}/2)^2+1/2\Big)\right]_0^1 + \frac{\sqrt{2}}{4}\left[\arctan\frac{x-\sqrt{2}/2}{\sqrt{2}/2}\right]_0^1 =$$

$$= -\frac{\sqrt{2}}{8}\ln(2-\sqrt{2}) + \frac{\sqrt{2}}{4}\Big(\arctan(-1+\sqrt{2}) + \frac{\pi}{4}\Big)$$

Now $\ln(2+\sqrt{2}) - \ln(2-\sqrt{2}) = \ln\frac{(2+\sqrt{2})}{(2-\sqrt{2})} = \ln\frac{\sqrt{2}+1}{\sqrt{2}-1} = \ln((\sqrt{2}+1)^2)$. You are certainly not expected to know that $\arctan(\sqrt{2}+1) = 3\pi/8$, $\arctan(\sqrt{2}-1) = \pi/8$, but these are true, and so the final result can be simplified to

$$\int_0^1 \frac{dx}{x^4+1} = \frac{\sqrt{2}}{4}\Big\{\ln(\sqrt{2}+1) + \pi/2\Big\}$$

(e)

$$\text{subst. } u = x^2: \quad \int_0^1 \frac{x\,dx}{x^4+1} = \left[\frac{1}{2}\arctan(x^2)\right]_0^1 = \frac{\arctan 1 - \arctan 0}{2} = \frac{\pi}{8}$$

(f)

$$\int_0^1 \frac{x+1}{x^2+x+1}\,dx = \int_0^1 \left(\frac{x+1/2}{(x+1/2)^2+3/4} + \frac{1/2}{(x+1/2)^2+3/4}\right)dx = \frac{1}{2}\ln 3 + \frac{\pi}{6\sqrt{3}}$$