— Some remarks on series — Jochen Denzler Mar 2001

Series, seriously

"Series" and "infinite series" means the same thing. "Finite series" would be as boring as a flat mountain.

You may view an infinite series as a(n ordered) sum of infinitely many terms, like eg.

$$1 + 2 + 3 + 4 + 5 + \cdots \qquad \text{(a bad example, but still an example)}$$

I have chosen this example, *because* it is a bad one: we cannot add up infinitely many terms. So, unlike the finite sum $1+2+3+4$, where we have a *result*, namely 10, there is no legitimate result for the infinite sum just given. (If you think, $+\infty$ should be a legitimate result, you have the right intuition, but be alerted that we have good reasons to discriminate against ∞ and that we will not accept ∞ as a result for a series.)

We say that a series $a_1 + a_2 + a_3 + a_4 + \cdots$ converges, if the *sequence* of its partial sums s_k converges. $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3, \ldots$. Then we call this limit the value (result) of our infinite sum. A warning is at hand here: In doing so, we change (slightly) the meaning of one of the most basic objects of mathematics, namely the meaning of *addition*, an operation with which you are familiar since primary school. Namely, it was originally *not* possible to add infinitely many terms, now it becomes possible in certain circumstances. If we fiddle around with something as fundamental as addition, we may be in for some (bad) surprises. Well-known rules for finite sums may cease to hold true for this new concept of a "'sum of infinitely many terms". And to make clear that there is something new to be discussed here, beyond the concept of a sum, we prefer to speak of series rather than sums of infinitely many terms.

We will only manipulate *convergent* series. In contrast, you should view divergent (i.e., not convergent) series as nothing more than a sequence of numbers separated by + signs rather than commas. Don't try to actually obey the '+' sign and to start adding terms of a divergent series! There is no use for divergent series in undergrad' mathematics. (They do have a use of their own in more advanced contexts.)

The purpose of these notes is to put it straight what is and what is not legitimate to do with series. Generally, series are well-behaved objects, if you know about the few restrictions.

1. If you have two convergent series $a_1 + a_2 + a_3 + \cdots = A$ and $b_1 + b_2 + b_3 + \cdots = B$, you may add them term by term as you would do with finite sums:
   $A + B = (a_1 + a_2 + a_3 + \cdots) + (b_1 + b_2 + b_3 + \cdots) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots$.
   Similar for subtracting them, and for multiplying them with a single expression:
   $k \cdot (a_1 + a_2 + a_3 + \cdots) = k \cdot a_1 + k \cdot a_2 + k \cdot a_3 + \cdots$.

2. However, watch out that, if you want to take a convergent series apart again into two series, you still have to make sure that the two series thus obtained are convergent.[1] Otherwise, the operation is not legitimate:
   $(\frac{1}{2} - \frac{1}{3}) + (\frac{1}{4} - \frac{1}{9}) + (\frac{1}{8} - \frac{1}{27}) + (\frac{1}{16} - \frac{1}{81}) + \cdots = (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots) - (\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots) \qquad \checkmark$
   is perfectly legitimate, *because* the resulting series converge. In contrast,
   $(1-1) + (1-1) + (1-1) + (1-1) + \cdots = (1+1+1+1+\cdots) - (1+1+1+1+\cdots) \qquad ↯$

[1] Similarly, you would not accept to calculate $\lim((-1)^n - (-1)^n)$ (which is clearly $\lim 0 = 0$) as $\lim(-1)^n - \lim(-1)^n$, because splitting must not be done in a way such as to create divergence.

is *not* legitimate, because the right hand side is $(undefined) - (undefined)$; don't think this is 0. Undefined minus undefined is still undefined. The left hand side however is clearly $0 + 0 + 0 + 0 + \cdots = 0$.

3. In particular this last example shows you that it is not legitimate to drop infinitely many parentheses from a convergent series (as you would easily do with finitely many in a finite sum):
   $(1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0$, but
   $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ is a *divergent* series. ↯
   However, you may always *introduce* parentheses in a *convergent* series:
   $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots$
   $= (1 + \frac{1}{2}) + (\frac{1}{4} + \frac{1}{8}) + (\frac{1}{16} + \frac{1}{32}) + (\frac{1}{64} + \frac{1}{128}) + \cdots =$
   $= \frac{3}{2} + \frac{3}{8} + \frac{3}{32} + \frac{3}{128} + \cdots$. ✓
   Note that setting parentheses in a series amounts to passing to a subsequence of the sequence of partial sums (i.e., kicking out some members of the sequence). In the example, without parentheses, the sequence of partial sums is $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32}, \frac{127}{64}, \frac{255}{128}, \ldots$. After setting the parentheses, you only retain every other term, namely $\frac{3}{2}, \frac{15}{8}, \frac{63}{32}, \frac{255}{128}, \ldots$, which does not destroy the convergence. However, dropping parentheses introduces more terms to be interspersed in the sequence of partial sums, and the extra members in this sequence can destroy its convergence.

4. You are very familiar with $a + b = b + a$. For infinite series, a caveat is to be observed. For convergent series with positive terms, no trouble arises. More generally, for *absolutely convergent* series, no trouble arises. (This latter concept will be dealt with in 8.5 of the textbook, so don't be surprised, if you don't know it yet.)

   However, when this extra assumption is not satisfied, then you should not change the order of infinitely many terms in the sum. Example 8 on p. 657 explains lucidly what may happen otherwise: you could get any desired result by appropriately rearranging terms in a convergent, but not absolutely convergent series.

   Luckily, you will (a) rarely have reason to rearrange terms in a way that may be prohibited and (b) mainly have to do with so called power series which would satisfy the condition of absolute convergence anyway. (Details see 8.6 of the textbook.)

5. The two previous restrictions (dropping parentheses and rearranging terms) are only significant, if you want to do it with infinitely many. You may drop finitely many parentheses and rearrange finitely many terms in an infinite series according to the rules you know for finite sums since elementary school. No bad surprises here. So:
   $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \cdots = 1 + \frac{1}{9} + \frac{1}{4} + \frac{1}{25} + \frac{1}{49} + \frac{1}{16} + \cdots$ ✓
   (two odd terms, one even term, etc.) is ok, because all terms are positive (or because the series is *absolutely* convergent. In contrast, the same rearrangement of terms is not legitimate in the following example:
   $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - + \cdots$ "=" $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + + - \cdots$ (wrong!). ↯
   Actually, the left side converges to $\ln 2$, whereas the right side converges to a different number that is not so easy to find and happens to be $\frac{3}{2} \ln 2$. However,
   $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - + \cdots = 1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - + \cdots$, ✓
   where only the first four terms have been rearranged, with no change after the fifth term, is legitimate again.

6. With $k \cdot (a_1 + a_2 + a_3 + \cdots) = k \cdot a_1 + k \cdot a_2 + k \cdot a_3 + \cdots$ under number 1, you have

legitimately used the distributive law. You may consider to use the distributive law for multiplying series, like
$(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots)\cdot(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots)$. If you try it, you will see, that you have to make up your mind in which order to arrange the resulting terms (any choice will be an arbitrary one). In cases where order matters (according to number 4) this procedure is therefore *not* legitimate. The good news is that for *absolutely* convergent series, using the distributive law *is* legitimate, and you may arrange the resulting terms in any order you like. If you think that, even when it is legitimate to expand parentheses that way, you'd rather not do it, because it's still a mess, you are right of course, and you will rarely (if ever) have to do this task in freshman calculus.

In short there are two possible trouble spots with series: dropping infinitely many parentheses may destroy convergence, and then you must not drop them; and rearranging infinitely many terms requires *absolute* convergence, otherwise it's not legitimate. Using the distributive law for multiplying series automatically involves rearrangement of infinitely many terms and is therefore subject to that same constraint. Otherwise convergent series behave like finite sums.

And here are a few surprising (advanced) results, without proof, just for viewing pleasure:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\ldots=\frac{\pi^2}{6}$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\ldots=\frac{\pi^4}{90}$$

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\frac{1}{6^3}+\ldots=???$$

(You get similar formulas for the reciprocals of even powers, but nobody has ever come up with a decent formula for the sum of the reciprocals of odd powers. It was shown only a few years ago that this latter number "???" is irrational!)

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-+\ldots=\ln 2$$

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-+\ldots=\frac{\pi}{4} \qquad (*)$$

You will come to understand the latter two soon, and possibly the following two as well.

$$1-\frac{1}{2\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 5}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 9}-+\cdots=\ln(1+\sqrt{2})$$

$$\frac{1}{2}-\frac{1}{2\cdot 4}+\frac{1\cdot 3}{2\cdot 4\cdot 6}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}-+\cdots=\sqrt{2}-1$$

Neither of these series are particularly well suited for numerical calculation of the right hand side, and the last four are actually very poor for that purpose: you have to add zillions of terms in (*) to get a modest number of digits for $\pi/4$.