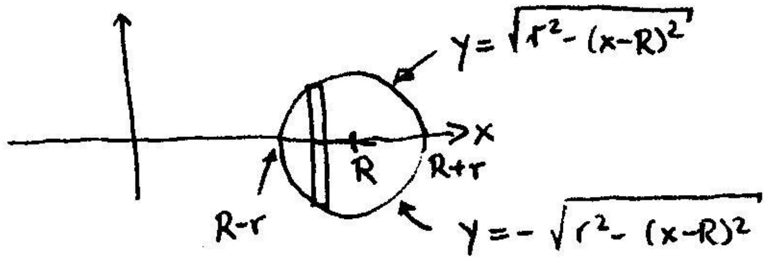
$= \pi^2 R r^2 + 2\pi \left[r^2 y - \frac{1}{3} y^3 \right]_0^r = \underline{\underline{\pi^2 R r^2 + \frac{4\pi}{3} r^3}}$ $\boxed{(\#25 \text{ cont'd})}$

Similarly $V_{inner} = \pi^2 R r^2 - \frac{4\pi}{3} r^3$

Second method for volume: shells



$$\text{Volume} = \int_{R-r}^{R+r} 2\pi x \cdot 2\sqrt{r^2 - (x-R)^2}\, dx \underset{x-R=t}{=} 4\pi \int_{-r}^{r} (t+R)\sqrt{r^2-t^2}\, dt$$

$$= 4\pi \underbrace{\int_{-r}^{r} t\sqrt{r^2-t^2}\, dt}_{=0 \text{ by symmetry}} + 4\pi R \underbrace{\int_{-r}^{r} \sqrt{r^2-t^2}\, dt}_{\frac{\pi}{2} r^2 \text{ (half disc, or: see below)}} = \underline{\underline{2\pi^2 R r^2}}$$

If we want the outer half only:

$$V_{outer} = 4\pi \int_0^r t\sqrt{r^2-t^2}\, dt + 4\pi R \underbrace{\int_0^r \sqrt{r^2-t^2}\, dt}_{\frac{\pi}{4} r^2}$$

subst. $r^2 - t^2 = u$

$t\,dt = -\frac{1}{2} du$

$$= 4\pi \int_{r^2}^{0} -\frac{1}{2}\sqrt{u}\, du + \pi^2 R r^2 = 2\pi \left[\frac{2}{3} u^{3/2} \right]_0^{r^2} + \pi^2 R r^2$$

$$= \underline{\frac{4\pi}{3} r^3 + \pi^2 R r^2}$$

Similarly $\underline{V_{inner} = -\frac{4\pi}{3} r^3 + \pi^2 R r^2}$