#40:

$$\int_0^\infty x^{t-1} e^{-x}\,dx = \underbrace{\left[\frac{1}{t} x^t e^{-x}\right]_0^\infty}_{=0} + \frac{1}{t}\int_0^\infty x^t e^{-x}\,dx$$

$$\uparrow \quad \downarrow$$

$$\frac{1}{t}x^t \quad -e^{-x}$$

(a) In other words: $\Gamma(t) = \frac{1}{t}\Gamma(t+1)$, or

$$\Gamma(t+1) = t\,\Gamma(t)$$

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty = 1$$

$$\Gamma(2) = 1\cdot\Gamma(1) = 1$$

$$\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1 = 2$$

$$\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2\cdot 1 = 6$$

etc.

$$\underline{\underline{\Gamma(n) = (n-1)!}}$$

(b) $$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\,dx \underset{\substack{x=y^2 \\ dx = 2y\,dy}}{=} 2\int_0^\infty e^{-y^2}\,dy \underset{\text{from \#39}}{=} \sqrt{\pi}$$

(c) $$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \tfrac{3}{4}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{7}{2}\right) = \tfrac{5}{2}\Gamma\left(\tfrac{5}{2}\right) = \tfrac{15}{8}\sqrt{\pi}$$