UTK – M231 – Differential Equations
Notes on Chapter 2 – Jochen Denzler, Aug/Sep 2002

Sec 2.1, 2.2: Separable equations

For certain first order ODEs, there is a method to find a solution formula: this method is called **separation of variables**, and the equations for which it works are called **separable**. You have seen the method already: p2 of the book (radioactive decay), or my example in the notes concerning implicit solutions. The method consists of bringing all y's (dependent variable) on one side, all x's (independent variable) on the other side of the equal sign, and then integrating.

Example $dv/dt = g - kv$ for an unknown function $v = v(t)$ with given constants g, k; the physical meaning of the example (fall with air resistance) has been explained in the book and the lecture, I needn't type it up here.

Of course, we will often write equations that describe moving particles as a 2nd order ODE for the location x, because we are interested in the position as well, not only in the velocity: namely $d^2x/dt^2 = g - k dx/dt$ in this example. Here we are lucky that x alone doesn't occur, only derivatives of x, and this is why we can write it as a first order equation for $v = dx/dt$, and even better, a first order eqn that fits in this chapter!

There are two ways of writing the separation of variables, and they are equivalent; understand them both, then take your choice:

easy, formal, a bit mystifying (using Leibniz's d/dt notation for derivatives). Hides the chain rule / integration by substitution behind the d's.

$\dfrac{dv}{dt} = g - kv$	separate var's, treating dv and dt as ordinary symbols
$\dfrac{dv}{g - kv} = dt$	write an integral sign
$\displaystyle\int \frac{dv}{g - kv} = \int dt$	

less mystifying, pedagogical (using Newton's $'$ notation for derivatives): Makes chain rule / integration by substitution explicit.

$v' = g - kv$	in separating variables, v' and v go together, every explicit t goes on the other side; but...
$\dfrac{v'}{g - kv} = 1$	here there was no explicit t, so we only got 1 on the rhs. Note that I said *explicit* t, because...
$\dfrac{v'(t)}{g - kv(t)} = 1$	... there is a hidden t, as v, v' are functions of t. Here I made this manifest. Now integrate wrt. t
$\displaystyle\int \frac{v'(t)}{g - kv(t)} dt = \int dt$	and use integration by substitution,

which brings us exactly back to the final formula of the method on the left

The pure ODE part of the task is done already! But there remains a calculus and an algebra part:
(1) calculus: evaluate the integrals; often possible, sometimes not. We see to it that it's possible in our examples. If you can't find antiderivatives: stop here and call it progress.
(2) algebra: Solve for the unknown function (alias dependent variable). Sometimes possible, sometimes not. If not, you're stuck with an implicit solution and also call it progress.

Note that we may have lost some solutions in the process of dividing (2nd line); namely if there was a solution for which $g - kv = 0$ (in this example, sure there is one: $v \equiv g/k$ is a solution) it fled in horror, when we put that term in the denominator.

General principle: If the right hand side $f(x, y)$ of $dy/dx = f(x, y)$ is actually a product of two terms, one of which depends only on x, the other only on y, i.e., if we can write $f(x, y) = g(x)p(y)$, then the ODE is separable, otherwise not. (But of course, an expression may not *look* like such a product at first glance, even it is one: for instance, e^{x+y}.)

Section 2.2 gives other examples.

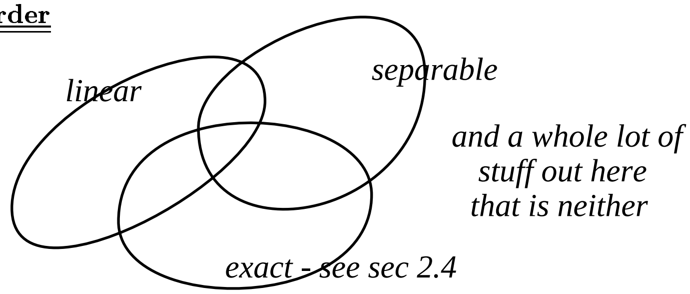
Hwk for 2.2: # 1–5, 7, 12, 17–19, 21; In 17–19, 21 also give the maximal interval of existence (*). 27b, 29, 32 a–c (**)

(*) Remember: If a solution to some ODE with initial condition eg., $y(1)$ =whatever, is a function that is not defined on the entire real line, like, eg., $y(x) = 1/x$, it is sensible to consider the solution only on the largest *interval* containing the initial point ($x = 1$ in the example).

(**) Comment: 32 is not an exceptional example, but such examples occur as normally as rainy days: Solutions may exist in a smaller interval than could be suspected by just eyeballing the ODE, and the precise interval of existence may depend on the initial data.

Sec 2.3: Linear ODEs of first order

One remark ahead: We are not sorting equations into drawers here: The types of ODEs about which you're learning here will *intersect*.



Be reminded of the definition of *linear* ODEs: An (O)DE is *linear*, if the unknown function (in other words: the dependent variable) occurs linearly in the equation, that is, if the dependent variable is y and the independent variable is x (the unknown function being $x \mapsto y(x)$), we have only $y, y', y'' \ldots$, possibly multiplied by given expressions of x, and added together.
So the eqn should NOT contain: y^2, y'^2, $\sin y$, $y \cdot y'$, $1/(1+y)$, ...
but ok are: x^2, $\sin x$, $e^x \cdot y$, $e^x y'/(1+x)$, y''' (also if written as d^3y/dx^3, some folks got confused on this one)...

This definition applies to all kinds of equations; the world "linear" always refers to the unknown, sought-for object in whatever type of eqn you have.

In this section, we restrict to linear ODEs of *first order*. For them, another "recipe for calculating a solution" exists. And this recipe has not so much to do with linearity alone, but is tuned to the precise situation (1) linear and (2) ODE and (3) first order. You'll learn about linear ODEs of higher order later, and in that context, the role of "linear" will become more lucid, not eclipsed by the presence of the other helpful features.

The recipe consists of multiplying the equation with an **integrating factor**, usually called $\mu(x)$, to bring it in a shape where you can just take integrals; hence the name integrating factor. Let me first illustrate what an integrating factor *is*, without revealing how to *find* the integrating factor; subsequently, I'll show you how to find it.

Example: $xy'(x) + y(x) = \sin x$. The lhs can be seen to be the derivative $\frac{d}{dx}(xy(x))$, so you can just integrate without first multiplying the equation. The integrating factor is 1. So you get

$$\frac{d}{dx}\Big(xy(x)\Big) = \sin x \qquad \text{hence} \qquad xy(x) = \int \sin x dx = -\cos x + C$$

Example: $xy'(x) - y(x) = \sin x$. You won't find a similar convenience this time. However, if you multiply the equation by $\mu(x) := 1/x^2$, you get $y'(x)/x - y(x)/x^2$ on the lhs, and this is indeed $\frac{d}{dx}(y(x)/x)$. So here you get

$$\frac{d}{dx}\frac{y(x)}{x} = \frac{\sin x}{x^2} \qquad \text{hence} \qquad \frac{y(x)}{x} = \int \frac{\sin x}{x^2} dx$$

The ODE part of the job is done (no derivatives of the unknown function left), and the fact that there is a calculus problem left, namely to evaluate $\int \frac{\sin x}{x^2} dx$, is a different issue (you may have to do this one numerically).

Next I need to resolve two mysteries: How to find the integrating factor (i.e., how to know with what expression to multiply the ODE), and how to come up with the ingenuity to recognize an expression like $xy'(x) + y(x)$ or $y'(x)/x - y(x)/x^2$ as $\frac{d}{dx}$ of some (and which?) other expression? Both questions are related and are answered at the same time:

Unlike the examples, let us start with a situation where there is no coefficient in front of y'; this can always be achieved by a simple division. So, assume we have to solve

$$y'(x) + P(x)y(x) = Q(x) \quad \text{with } P(x), Q(x) \text{ certain given expressions in the independent variable} \tag{1}$$

We want to multiply the eqn with a certain $\mu(x)$, yet to be determined, with the purpose of receiving an expression $\frac{d}{dx}$(something explicit) on the lhs. So we'll get

$$\mu(x)y'(x) + \mu(x)P(x)y(x) = \mu(x)Q(x) \tag{1a}$$

and want to choose μ such that

$$\mu(x)y'(x) + \mu(x)P(x)y(x) = \frac{d}{dx}(???)$$

Clearly the '???' should contain $y(x)$, otherwise: where else could we get the $y'(x)$ from? The examples we have seen suggest

$$??? = (\text{expression in } x \text{ alone})y(x)$$

and trying this we see that the 'expression in x alone' has to be $\mu(x)$. So we want to find μ in such a way that

$$\mu(x)y'(x) + \mu(x)P(x)y(x) \equiv \frac{d}{dx}(\mu(x)y(x)) \tag{2}$$

and this requires $\mu' = P\mu$: We again have to solve a linear ODE just to find μ! But this is an easier one, because it is separable. It's solution is $\mu(x) = C_0 \exp\int P(x)dx$, and we are happy with *some* solution, so we can just as well choose $C_0 = 1$.

At this stage, we have answered both questions, namely

> *How to find the integrating factor (i.e., how to know with what expression to multiply the ODE), and how to come up with the ingenuity to recognize an expression like $xy'(x) + y(x)$ or $y'(x)/x - y(x)/x^2$ as $\frac{d}{dx}$ of some (and which?) other expression?*

μ is found by solving the separable linear 1st order ODE $\mu' = P\mu$, and it is $\mu(x) = \exp\int P(x)dx$. And then eqn (2) tells us how to rewrite the lhs of the original equation, to which we now return:

Its form (1a) can now be written as

$$\frac{d}{dx}(\mu(x)y(x)) = \mu(x)Q(x) \tag{1b}$$

and we just integrate:

$$\mu(x)y(x) = \int \mu(x)Q(x)\,dx + C \tag{3$_{\text{gen}}$}$$

This completes the ODE part of the job. As always, you may or may not find it possible to do the calculus part, namely evaluating the integral on the rhs; and you may already have had calculus trouble in explicitly carrying out $\int P(x)\,dx$ when determining μ. If this happens, the unevaluated integrals remain in the solution formula.

If you have an IVP, say (1) together with $y(2.37) = 17.92$, you can determine the constant of integration after evaluating (3). BUT you can ALSO determine it right at the stage of (3), before evaluating integrals. If you cannot evaluate the integral, this is the only choice: From (1b) to (3), you do *definite* rather than indefinite integrals and get

$$\mu(x)y(x) - \mu(2.37)\overbrace{y(2.37)}^{=17.92} = \int_{2.37}^{x} \mu(t)Q(t)\,dt \tag{3$_{\text{IVP}}$}$$

Hwk for 2.3: # 1–6, 8, 11, 16–18, 20, 25a, 28

Organization:

We postpone (possibly omit) 2.4, which is essentially a jazzed-up footnote to multivariable calculus, and 2.5.

The grading procedure will slightly change: the time allocated for the grader is not sufficient to grade all hwk, if grading is defined to include diagnosing difficulties and give useful comments. I will therefore select problems for grading, not disclosing beforehand which will be graded. With a priority for helpful instruction in mind, I will preferably choose for grading those problems that cause difficulties. Should this strategy appear to make a high hwk percentage unduly difficult, I'll make appropriate allowances.

If you include questions in your hwk (eg., I have noticed one student's question "Why don't we evaluate the integral?" in 2.2#27) and you ask them in a clearly visible way so they cannot be overlooked (maybe mark them with a color question mark), they will receive an answer independent of whether they concern graded or ungraded homework.

As difficulties and/or mistakes are often shared by a number of students, time will be saved by not putting the same comment everywhere, but on a "generic comment sheet" (GCS) that will be xeroxed for the class. So if you find (beginning with 2.2) a remark like →GCS on your graded hwk, a comment applying to your solution will be found there.

An important integrating factor in mechanics:

This material is not in the book, but I'll make it part of the course, and it may be in the exam.

If x describes the position of a particle (as a function of time t), then $\ddot{x}$ is the acceleration, and $m\ddot{x}$ the force causing the acceleration, according to Newton's law of motion. Suppose this force depends only on the position of th particle, but not on its velocity, nor explicitly on the time. We get an ODE of 2nd order: $m\ddot{x} = F(x)$. For instance, for a pendulum, where a mass is suspended at a twine (or a thin rod) of length ℓ, with x denoting the angle from the vertical, we have $F(x) = -m\frac{g}{\ell}\sin x$.

Remember that an integrating factor is a factor by which to multiply an ODE such that one can then take integrals easily. The method of integrating factors is not restricted to 1st order linear ODEs. Rather, for 1st order linear ODEs, the method can be used routinely, wheras in general, it requires hindsight or ingenuity. The example in this little section represents another problem class in which this hindsight and ingenuity has turned into a routinely applicable procedure.

Principle: in the class of ODEs specified in the first paragraph, the velocity $\dot{x}$ can be used as an integrating factor, reducing the 2nd order ODE to a 1st order ODE (even a separable one). The constant of integration you get from integrating the equations of motion is the energy. (And for this reason it will be denoted as E rather than C below.)

$$\begin{aligned} m\ddot{x}(t) &= -\frac{mg}{\ell}\sin x(t) \\ m\dot{x}(t)\ddot{x}(t) &= -\frac{mg}{\ell}\sin x(t)\,\dot{x}(t) \\ \frac{d}{dt}\left(\frac{1}{2}m\dot{x}(t)^2\right) &= \frac{d}{dt}\left(\frac{mg}{\ell}\cos x(t)\right) \\ \frac{1}{2}m\dot{x}(t)^2 &= \frac{mg}{\ell}\cos x(t) + \mathrm{E} \end{aligned}$$

If you want another training example for separable equations, you may now finish solving this ODE. Be aware however, that you'll get an integral that you cannot explicitly evaluate.

2.6: Substitutions and Transformations

The general 1st order ODE is

$$y' = \text{any expression in terms of } x,\ y$$

(Well, almost general: we assume that you can algebraically solve for y'. Stuff like $y' + \sin y' = x + y$ isn't much fun and not too important either.) Special solution strategies exist for special kinds of expressions on the right hand side. You have seen two startegies already:

(1) If the rhs is a product of an expression in x alone and another expression in y alone, $y' = g(x)p(y)$, then the equation is separable.

(2) If the rhs is linear in y, namely $y' = -P(x)y + Q(x)$, then the integrating factor strategy applies.

Now you learn three more strategies:

(3) If the rhs can be expressed as a function of y/x alone, $y' = G(y/x)$, then the ODE is called **homogeneous**, and you substitute $v = y/x$ (consequently $y' = (xv)' = v + xv'$). The resulting ODE for the unknown function v is separable. Once you have found v, you get y trivially as $y = xv$.

(4) If the rhs can be expressed as a function of $ax + by$ alone, for some numbers a and b, as e.g., in $y' = (2x+y)^2 - (2x+y)$, use the substitution $z = ax+by$ and consequently $y' = (z' - a)/b$. The resulting ODE for the new unknown function z is separable, and once you have found z, you get y trivially.

(5) If the rhs is of the form $-P(x)y + Q(x)y^n$ for any real number $n \neq 0, 1$, then you substitute $v = y^{1-n}$. This will lead to a linear equation for v. This type is called Bernoulli equation.

I am omitting another strategy to be found at the end of Sec. 2.6.

Hwk for 2.6: # 10, 13, 18, 21, 22

Hwk from Review Problems p. 87f: # 1, 2, 4, 8, 9, 15, 17, 32. With 32, give the maximum interval of existence

non-Hwk: As we have omitted some of the material in Chapter 2, the Review Problems on p. 87 will include some that you are not expected to be able to solve. You may try to look at each of these problems and check which methods should apply; so in principle you should be able to determine for which of them you have no method available. I have come up with a list of the ones which I think you cannot be expected to do. I'll release this list soon on the course web site. (Only there, and on a separate page, so you may choose not to know it as long as you prefer to figure it out for yourself.)