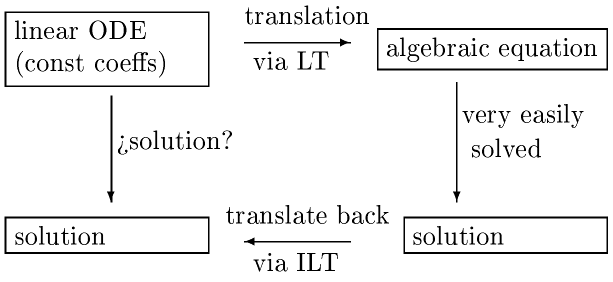
UTK – M231 – Differential Equations

Notes Chapter 7: The Laplace Transform and its Use for Solving ODEs – Jochen Denzler, Nov 2002

The method of Laplace transforms, to be explained below, is applicable only to linear ODEs; more specifically, to linear ODEs with constant coefficients. (There are a few cases where the LT method can be used for variable coefficients, but we won't dwell on them.) Therefore, this method will hardly expand the collection of ODEs you can solve. You will see soon, why it is nevertheless useful (often faster), and what its other advantages are.



The method consists of transforming the ordinary differential equation into an algebraic equation. You can view this as a translation process. The algebraic equation thus obtained is actually very easy to solve. The solution of the algebraic equation needs to be transformed back into a solution of the differential equation. This last step (Inverse Laplace Transform) is actually the most difficult and subtle one.

You have already seen one example of "translating an ODE into algebraic equation": For instance, instead of the ODE $y'' - 3y' + 2y = 0$, you solved the algebraic equation $m^2 - 3m + 2 = 0$, and its solution "$m = 1$ or $m = 2$" is translated back into a solution of the ODE $y = c_1 e^x + c_2 e^{2x}$. The LT method is a different kind of translation. Before abording it in detail, let me give you a list of inhomogeneities for which this method is useful:

- Polynomials, trig functions (sin, cos), exponentials and products thereof (not exciting: you can do these with undetermined coefficients already)
- step functions, and other piecewise defined functions (even if you can handle these "piece by piece" without LT, this approach would become unduly tedious)
- periodic functions, even if they are not pure sines and cosines, like . Such functions are important in electronics, where they represent special AC voltages.
- short pulses.

The Laplace transform of a function $f = f(t)$ (t means usually time) is a function of a new variable s, denoted as $\mathcal{L}\{f\}(s)$, or $\mathcal{L}\{f(t)\}(s)$, and defined by

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st}\,dt$$

The word Laplace transform is used in two meanings: (a) the method or procedure $\mathcal{L}$ that takes a function f and produces another function $\mathcal{L}\{f\}$ from it; (b) the output of this procedure, namely $\mathcal{L}\{f\}$.

I'll omit typesetting examples here; you have done them as homework. We will sometimes adopt the convention to denote the Laplace transform of a function by the corresponding capital letter: $\mathcal{L}\{f\}(s) =: F(s)$. The notation with the t in it ($\mathcal{L}\{f(t)\}(s)$, rather than $\mathcal{L}\{f\}(s)$) is actually a disgrace, because the Laplace transformed function does *not* depend on t anymore; in it, the t is a dummy variable that could be renamed arbitrarily, like you can rename a variable of integration. Nevertheless, I will often use this disgraceful notation, for the following reason: It would be no problem to write $\mathcal{L}\{\sin\}(s)$ instead of the disgraceful $\mathcal{L}\{\sin t\}(s)$, but how would you get rid of the t without ambiguity in $\mathcal{L}\{e^{-2t+1}\sin(t-\frac{\pi}{5})\}(s)$?

From this definition, you can obtain the following fundamental properties of the LT:

- Linearity: $\mathcal{L}\{\alpha f(t) + \beta g(t)\}(s) = \alpha\mathcal{L}\{f(t)\}(s) + \beta\mathcal{L}\{g(t)\}(s)$
- Translation in s: $\mathcal{L}\{e^{\alpha t}f(t)\}(s) = \mathcal{L}\{f(t)\}(s-\alpha)$.
- LT of Derivatives: $\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0)$. (This rule is obtained using integration by parts.)
  Higher order derivatives: $\mathcal{L}\{f^{(n)}(t)\}(s) = s^n\mathcal{L}\{f(t)\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$.
- Derivative of the LT: $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}(s)$.

The LT of derivatives is the most significant property when we deal with ODEs: Consider the IVP

$$y''(t) - 3y'(t) + 2y(t) = 0\ , \quad y(0) = 1\ , \quad y'(0) = 2$$

We know from the general theory that this IVP has exactly one solution $y(t)$; in order to find this solution, we call its Laplace transform $Y(s)$, obtain an equation for $Y(s)$, which we then solve; and finally reconstruct $y(t)$ from its Laplace transform $Y(s)$. By taking the LT of both sides of equation, using the above properties of the LT, we get

$$s^2Y(s) - sy(0) - y'(0) - 3\Big(sY(s) - y(0)\Big) + 2Y(s) = 0$$

or, using the initial conditions,

$$s^2Y(s) - s - 2 - 3\Big(sY(s) - 1\Big) + 2Y(s) = 0\,, \quad \text{hence} \quad (s^2 - 3s + 2)Y(s) = s - 1$$

It is indeed VERY easy to solve this equation, namely

$$Y(s) = \frac{s-1}{s^2 - 3s + 2} = \frac{1}{s-2}$$

The cancellation in the last step is a lucky coincidence, due to a particularly convenient choice of initial data. The polynomial $s^2 - 3s + 2$, by which $Y(s)$ gets multiplied, is the same polynomial that enters into the characteristic (alias auxiliary) equation, when you use the method from chapter 4. The only question that remains is: Whose LT is $\frac{1}{s-2}$? As you have no formula for the inverse Laplace transform, this is a difficult question, unless you see the answer, by recognizing $\frac{1}{s-2}$ as the Laplace transform of e^{-2t}. But once you notice this, you have the solution $y(t) = e^{-2t}$. The reasoning in this step deserves a closer scrutiny, which will follow in a moment.

But first, note a few peculiarities of the method:
(1) Unlike the methods of chapter 4, the LT method uses the initial values from the very beginning, rather than storing them untouched until the end, after the general solution is found: so you find the solution of the IVP without the detour through finding the general solution of the ODE.
(2) Moreover, you do not need to split an inhomogeneous ODE into its homogeneous part and an inhomogeneity. If our IVP had been $y'' - 3y' + 2y = \sin t\,,\ y(0) = 1\,,\ y'(0) = 2$, the LT would just as quickly have produced the equation $s^2Y(s) - s - 2 - 3\Big(sY(s) - 1\Big) + 2Y(s) = \frac{1}{s^2+1}$. In this example, the question "Whose LT is the $Y(s)$ obtained?" would be more difficult to answer at the present stage. But this difficulty (which we will overcome soon anyway) does not deduct from the fact that the flow diagram of the solution procedure via LT is the same, whether the ODE is homogeneous or inhomogeneous.

Now let us scrutinize the problem of finding $y(t)$ from $Y(s)$: I'll present the basic theorems in Q&A form:

Question: Given any expression $F(s)$, is there always an inverse Laplace transform (ILT), i.e., a function $f(t)$ such that $\mathcal{L}\{f(t)\}(s)$ equals our given $F(s)$?
Answer: Definitely NO! — For instance, $\sin s$ is nobody's Laplace transform. Indeed, for any $f(t)$, $\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt \to 0$ as $s \to \infty$, whereas $\sin s \not\to 0$ as $s \to \infty$. Let me explain briefly why $\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt \to 0$ as $s \to \infty$: Basically, because $e^{-st} \to 0$ as $s \to \infty$, for every $t > 0$, so we end up with $\int_0^\infty 0 \cdot f(t)\,dt$ in the limit $s \to \infty$. This explanation is NOT complete and lacks one key ingredient needed for a mathematically compelling argument; but for the purpose of this course, it's good enough for the moment. — Another example, $F(s) = e^{-s}\sin s$ is also nobody's LT. This is due to more advanced properties of the LT that we cannot explain here. — Yet another example $F(s) = e^{-|s-1|}$ is nobody's LT either. Here, the reason is that any $\mathcal{L}\{f(t)\}(s)$ is a smooth function of s, unlike $e^{-|s-1|}$. This is another property of the LT, and it is due to the fact that e^{-st} is a smooth function of s, and e^{-st} is the only occurrence of s in $\int_0^\infty f(t)e^{-st}\,dt$. (Again a key ingredient towards a mathematically compelling argument is missing here, but we'll call it good enough for the purpose of this course and for the moment.) But note, from your homework examples, that even if $f(t)$ is discontinuous in t, its Laplace transform $F(s)$ will be smooth in s.

Question: Does this fact, namely that for given $F(s)$ there may not exist an ILT $f(t)$, cause us problems, when we are seeking solutions $y(t)$ to an IVP for an ODE, and have to determine $y(t)$ from $Y(s)$?
Answer: No! Because we know such an IVP does have a solution $y(t)$, so the $Y(s)$ we obtain when trying to find $y(t)$ by means of the LT *is* the LT of somebody, namely of $y(t)$.

Question: Given $Y(s)$, and knowing there is some $y(t)$ (namely the yet unknown solution of an IVP) out there whose LT is $Y(s)$, could there be another function $\tilde{y}(t)$ with the same LT $Y(s)$? In other words, can we be sure that, once we find indeed an ILT to $Y(s)$, that this is indeed a solution to our IVP and we don't retrieve a counterfeit solution?

Answer: Yes and no: If you are picky, you have to admit that functions $y(t)$ and $\tilde{y}(t)$ that differ only in a few discrete points share the same LT $Y(s)$, because the integral defining $Y(s)$ wouldn't see these few points. However, there can be only one *continuous* $y(t)$ whose LT is $Y(s)$, and you would probably not even think of altering its values at a few discrete points t_i arbitrarily, just to get competing $\tilde{y}(t)$ with the same LT. When you are looking for solutions $y(t)$ of an ODE, they should be continuous for sure (they even need to have derivatives). And with this in mind, you can rely that selecting the continuous $y(t)$ does give you the correct solution; the only conceivable counterfeit solutions would give away their counterfeit nature by being discontinuous.

Remark: To help you appreciate the problem of counterfeit solutions just addressed, let me remind you of a high school (or middle school?) example where such counterfeit solutions do occur: In order to solve the (algebraic) equation $x+1=\sqrt{x+13}$, you first square the equation, then find two solutions $x=3$ and $x=-4$ from the resulting quadratic equation. However, the second of them, namely $x=-4$ does *not* solve the original equation $x+1=\sqrt{x+13}$, but rather solves $x+1=-\sqrt{x+13}$. It is a counterfeit solution that sneaked in in the first step, when you squared the equation. This is why they (hopefully) told you that you have to check your results by plugging them back into the original equation. Squaring an equation can introduce counterfeit solutions, because the *inverse* process, namely answering the question "Whose square is this given number (eg. 100)?" involves ambiguity. There are two possibilities, 10 or -10. — We had never discussed before whether taking the Laplace transform of an equation can introduce counterfeit solutions. It could, if the inverse Laplace transform were ambiguous. But if we consider continuous functions $y(t)$ alone, there is no such ambiguity, and therefore there are no counterfeit solutions.

Remark: There is however an ambiguity if *all* $f(t)$ whose Laplace transform is $F(s)$ have discontinuities (e.g., jumps). This does not concern solutions of an ODE, but it does concern right hand sides (inhomogeneities). We will often not specify the value of f at a jump, like $f(t)=\begin{cases} t & \text{if } t<2 \\ 0 & \text{if } t>2 \end{cases}$, leaving the value for $t=2$ unspecified. $\mathcal{L}\{f(t)\}(s)$ does not depend on this ambiguity. And neither does the solution of an IVP whose right hand side is $f(t)$. So we would adopt a physics oriented point of view, saying "who cares about the value of f at one single t?", and "all the choices $f(t)$ for an ILT of $F(s)$ are <u>essentially</u> the same"

Remark: One more caveat: Think of all the functions $f(t)$ or $y(t)$ whose LT you take, as defined *only* for $t>0$, even if their formula makes sense for $t<0$. This is simply because of the fact that the LT doesn't "see" negative t: When calculating $\mathcal{L}\{f(t)\}(s)$, you have an integral from $t=0$ to $t\to\infty$. So, when I said above that there is only one continuous $y(t)$ whose LT is $F(s)$, I intended to say "only one continuous $y(t)$" underlinedefined for $t>0$".

When you need to find the ILT of a given function $Y(s)$, you encounter the fact, that there is no formula available for this job. [In reality, there is a formula, but you wouldn't be able to understand, let alone use it, until you have attended the complex variable course (443); there are actually several, very different looking formulas, but only the complex variable formula is useful for practical calculation.] In theory, and full generality, doing inverse Laplace transforms is a very sophisticated job. In practice however, for the sole purposes you need here, it is merely a combination of algebra and looking up in tables.

When you have to take the inverse Laplace transform of a *rational* function $Y(s)$, you decompose $Y(s)$ into partial fractions, in the same way as you would have to do it, if you wanted to find $\int Y(s)\,ds$. I don't do details here, but refer you to calculus (or my calculus notes on the subject). All possible terms arising in a partial fraction decomposition can be inverse-Laplace-transformed separately, according to table 7.1 of the book. (That's not quite true; this particular table doesn't contain stuff like $\frac{1}{(s^2+a^2)^n}$; but these terms occur rarely, and I could give you an enhanced table containing such entries if you ever need it.)

If you can only do the ILT of rational functions, you can solve only those IVPs that you could already do with undetermined coefficients. But I told you that one of the good uses for the LT method is inhomogeneities in the ODE that are piecewise defined functions. In this case, you end up with functions $F(s)=e^{-as}\times$ a rational function, and you need to find their ILT. This is our next goal.

To this end, let us consider the following example:

$$y'' + y = g(t) := \begin{cases} 1 & \text{if } t \le 2 \\ e^{-(t-2)} & \text{if } t > 2 \end{cases} \qquad , \; y(0) = 1 \,, \; y'(0) = 1$$

I will first show you (in sketch only) how you could solve this problem without the use of LT, so that you can appreciate the benefits of the LT.

> The IVP splits into two separate problems without ifs: The first is just $y'' + y = 1\,,\ y(0) = 1, y'(0) = 1$ for $t \le 2$. You have solved this type of problem already: undetermined coefficients, or variation of parameters, or what you know about the Laplace transform already, does the job. The solution of the IVP turns out to be $y(t) = 1 + \sin t$. But you only need it for $t \le 2$, because for $t > 2$, we are talking about an entirely different equation, namely the equation $y'' + y = e^{-(t-2)}$. What initial values do we take for this second equation? Clearly, whatever we get at $t = 2$ from the solution to the first equation: namely $y(2) = 1 + \sin 2$ and $y'(2) = \cos 2$. Solving the second ODE with these initial values is also a straightforward task that can be achieved by any of the methods mentioned for the first ODE. The solution is $y = \frac{1}{2}(e^{-(t-2)} + \cos(t-2) + \sin(t-2) + 2\sin t)$. It only applies for $t \ge 2$, because for $t < 2$ we needed to solve the other differential equation. We can put the two pieces of solution together again and obtain the following:
>
> $$y(t) = \begin{cases} 1 + \sin t & \text{if } t \le 2 \\ \frac{1}{2}\Big(e^{-(t-2)} + \cos(t-2) + \sin(t-2) + 2\sin t\Big) & \text{if } t \ge 2 \end{cases}$$
>
> If you actually do the calculation, you may get the formula for $t > 2$ in a somewhat different form, but it would be equivalent through a trig' identity. I wrote it in this form, because I didn't want crazy numbers like $\cos 2$ to float around, but rather have them absorbed, where conveniently possible, in a combination like $\cos(t-2)$: such a term still displays a visible trace of the change happening at $t = 2$.

This little excursion shows that the advantage of the LT does not lie in doing things that were absolutely impossible before. But if you think of an example with many more cases than the two $t \le 2$ and $t > 2$, it becomes clear that you would have to solve many ODEs separately in turn, if you wanted to employ the above method. The LT method will turn out to be more convenient and more clearly organized. Because now, we start over with the same IVP, but this time using the LT method consistently.

You can already start the LT method on our example ODE with your present knowledge: You need to calculate $\mathcal{L}\{g\}(s)$, and the ifs in the definition of g will be handled by splitting the integral:

$$\mathcal{L}\{g(t)\}(s) = \int_0^2 1 \cdot e^{-st}\, dt + \int_2^\infty e^{-(t-2)} e^{-st}\, dt = \ldots = \frac{1 - e^{-2s}}{s} + \frac{e^{-2s}}{1+s} = \frac{1}{s} + e^{-2s}\left(\frac{1}{1+s} - \frac{1}{s}\right)$$

Even though I have left the evaluation of the integrals as a little hwk for you, I have done some other little job meticulously here: namely, I have organized the result by grouping those terms containing e^{-2s} together, and also those that do not contain e^{-2s}, separately (well, there wasn't much to be grouped for these latter). This grouping makes sense, because the terms with e^{-2s} are exactly those that trace the effect of the change that happened at $t = 2$.

Therefore, $y'' + y = g(t)\,,\ y(0) = 1\,,\ y'(0) = 1$ is transformed into

$$s^2 Y(s) - s - 1 + Y(s) = \frac{1}{s} + e^{-2s}\left(\frac{1}{1+s} - \frac{1}{s}\right)$$

with $Y(s)$ the LT of our yet unknown solution $y(t)$. Again, it is incredibly easy (precalculus) to solve this equation for $Y(s)$:

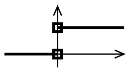
$$Y(s) = \frac{1}{1+s^2}\left(1 + s + \frac{1}{s} + e^{-2s}\left(\frac{1}{1+s} - \frac{1}{s}\right)\right) = \frac{1}{s^2+1} + \frac{1}{s} - e^{-2s}\left(\frac{1}{s} - \frac{1/2}{s+1} - \frac{1}{2}\frac{s+1}{s^2+1}\right)$$

Again, I have grouped terms according to the exponentials they contain, and I have already performed a partial fraction decomposition of all rational functions that have come up.

This is the moment where we really need new theory, which will help us answer the question "Whose Laplace transform is this?".

To get a convenient formalism, we have to understand 'how to get the ifs into a formula'. This is done, essentially, by giving a name to one particular function that has hidden an 'if' in its belly, and to study its properties. This function is called the Heaviside function, or unit step function,

$$u(t) := \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$



(We don't bother to define its value for $t = 0$; if you don't like this policy, you may take your own choice in defining it, but none of our results will be affected by the choice you make.) We can express all piecewise defined functions using the Heaviside function. For instance, our g from above:

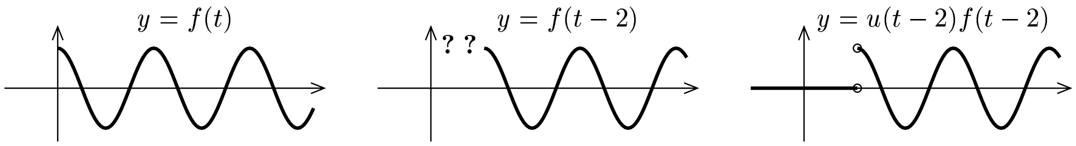
$$g(t) = \left\{ \begin{array}{ll} 1 & \text{if } t \leq 2 \\ e^{-(t-2)} & \text{if } t > 2 \end{array} \right\} = 1 + (e^{-(t-2)} - 1) \cdot u(t-2)$$

To display the structure of what's happening more clearly, let me do it with a more complicated example:

$$h(t) := \begin{cases} t & \text{if} \quad t < 2 \\ \sin^2 t & \text{if } 2 < t < \pi \\ e^t & \text{if } \pi < t < 5 \\ \frac{t+1}{t^4+t^2+4} & \text{if } 5 < t \end{cases} \qquad \text{can be rewritten as} \qquad \begin{aligned} h(t) = t &+ \\ &(\sin^2 t - t)\, u(t-2) + \\ &(e^t - \sin^2 t)\, u(t-\pi) + \\ &(\tfrac{t+1}{t^4+t^2+4} - e^t)\, u(t-5) \end{aligned}$$

(check it).

The Laplace transform of u is $\mathcal{L}\{u\}(s) = \frac{1}{s}$, the same as the LT of the constant 1, because, after all, u and the constant 1 coinside for $t > 0$, and this is all the Laplace transform sees. Given this, why is u helpful at all? The answer is that u becomes interesting when we shift the t variable. You have already had a formula about shifting the s variable, namely it was $\mathcal{L}\{f(t)e^{\alpha t}\}(s) = \mathcal{L}\{f(t)\}(s-\alpha)$. You will now learn a similar formula that shifts the t variable. A little bit of trouble is involved, because the LT does not see the "past" ($t < 0$). So if you try to shift a function like $f(t) = \cos t$ (understood to be defined *only* for $t \geq 0$, thus adapting the limited viewpoint of the Laplace transform), you face the problem of recovering a history "that is simply not there" (first two figures):



Multiplying with a correspondingly shifted Heaviside function makes it clear that we want to have 0 filled in for the gap, as seen in the 3rd of these figures.

With the Heaviside function, we can rewrite the Laplace transform slightly:

$$\mathcal{L}\{g(t)\}(s) := \int_0^\infty g(t)e^{-st}\, dt = \int_{-\infty}^\infty u(t)g(t)e^{-st}\, dt$$

Now it is the Heaviside function that takes care of the actual lower limit of integration, instead of an explicitly written lower limit 0. This may look like a trivial play, but it has the advantage that we needn't fiddle around with the limits of integration when we shift the t variable. Let $\alpha > 0$. Then

$$\begin{aligned} \mathcal{L}\{u(t-\alpha)f(t-\alpha)\}(s) &= \int_0^\infty u(t-\alpha)f(t-\alpha)e^{-st}\, dt = \int_{-\infty}^\infty u(t-\alpha)f(t-\alpha)e^{-st}\, dt = \\ &= \int_{-\infty}^\infty u(\tau)f(\tau)e^{-s(\tau+\alpha)}\, d\tau = e^{-\alpha s}\int_0^\infty f(\tau)e^{-s\tau}\, d\tau = e^{-\alpha s}\mathcal{L}\{f(t)\}(s) \end{aligned}$$

So here we have found our shift rule for the Laplace transform. It implies a corresponding rule for the inverse Laplace transform immediately:

$$\mathcal{L}\{u(t-\alpha)f(t-\alpha)\}(s) = e^{-\alpha s}\mathcal{L}\{f(t)\}(s) \quad , \qquad \mathcal{L}^{-1}\{e^{-\alpha s}F(s)\}(t) = u(t-\alpha)\mathcal{L}^{-1}\{F(s)\}(t-\alpha)$$

This formula gives us two advantages: (1) a faster (and more organized) way of calculating Laplace transforms of piecewise defined functions, and (2) a way to find inverse Laplace transforms of functions containing exponentials, like the one obtained above.

As an example for (1), we can redo our evaluation of $\mathcal{L}\{g(t)\}(s)$ from page 4:

$$g(t) = 1 + (e^{-(t-2)} - 1) \cdot u(t-2)\,, \text{ hence } \mathcal{L}\{g(t)\}(s) = \frac{1}{s} + e^{-2s}\mathcal{L}\{e^{-t} - 1\}(s) = \frac{1}{s} + e^{-2s}\left(\frac{1}{s+1} - \frac{1}{s}\right)$$

As an example for (2), we can now finish our solution of the IVP, by inverse-Laplace-transforming the $Y(s)$ found on page 4, namely

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{s} - e^{-2s}\left(\frac{1}{s} - \frac{1/2}{s+1} - \frac{1}{2}\,\frac{s+1}{s^2+1}\right)$$

Namely, we conclude that

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) &= \sin t + 1 - u(t-2)\mathcal{L}^{-1}\Big\{\frac{1}{s} - \frac{1/2}{s+1} - \frac{1}{2}\,\frac{s+1}{s^2+1}\Big\}(t-2) = \\ &= \sin t + 1 - u(t-2)\left(1 - \frac{1}{2}e^{-(t-2)} - \frac{1}{2}\Big(\cos(t-2) + \sin(t-2)\Big)\right) = \\ &= \begin{cases} 1 + \sin t & \text{if } t < 2 \\ \frac{1}{2}\left(e^{-(t-2)} + \cos(t-2) + \sin(t-2) + 2\sin t\right) & \text{if } t > 2 \end{cases} \end{aligned}$$

This result indeed coincides with the one we found previously by means of solving the ODE interval by interval.

The method can deal with piecewise defined functions that contain infinitly many pieces. This happens typically with periodic functions defined piecewise, like for instance the zigzag function whose graph you find on page 1. Alas, we are lacking one class for the details, but if you ever need it, this will be just within your reach, from what you have learned already.

THE END