Partial Fraction Decomposition (PFD)

These notes teach the use of PFD for two purposes, antiderivatives and inverse Laplace transforms, and will therefore be of interest at lesat at three levels: freshman calculus, sophomore ODE, and beyond (having a second look at the material). This multi-level audience is intentional: I hope that students keep and re-use this material and get directed into viewing the PFD as a multi-purpose tool from the very onset.

I use the following font distinctions:
- Broad overview in this present (Helvetica style) font.
- Basic techniques for everybody in this font (Times style)
- *Stuff specific to Inverse Laplace Transform in oblique font; should be ignored by freshman readers*
- **Boldface and underline for emphasis**

Use of complex numbers is encouraged thoughout and the corresponding material will be typeset in gray rather than black (or color, if you have a color rendering device); omitting complex number material is NOT encouraged, but is made possible due to the highlighting. Explanations are designed to require just the bare minimum of prereq's on complex numbers, but to teach all that is needed to use complex numbers fearlessly wherever convenient.

Primarily, partial fraction decomposition is a method of algebraically transforming rational functions into a certain standard form. This standard form characterizes the rational function according to the question: Where does it 'do bad things' (like going to infinity)? Partial fraction decomposition is a way of 'fingerprinting' rational functions for the purpose of **FBI** (Function Behavior Investigation). It is for this reason that it is useful, e.g., in finding antiderivatives, or inverse Laplace transfroms of a function. The idea that such 'fingerprints' identify the function stems from complex variables. Without this background you may justly find it weird, but it's nevertheless a deep truth.

An example of a partial fraction decomposition is:

$$\frac{x^3+x-3}{x^4+x^3-x^2+x-2} = \frac{x^3+x-3}{(x-1)(x+2)(x^2+1)} = \frac{-1/6}{x-1} + \frac{13/15}{x+2} + \frac{\frac{3}{10}(x+3)}{x^2+1} \tag{1}$$

The main task and main work will consist of understanding why we write it this way, and how we can find this form. Once you have written the expression in this way, it can be integrated by doing standard integrals, *and likewise the inverse Laplace transform (ILT) can also be done by referring to standard ILT's from the table.* Namely,

$$\int \left(\frac{-1/6}{x-1} + \frac{13/15}{x+2} + \frac{\frac{3}{10}(x+3)}{x^2+1}\right) dx = -\frac{1}{6}\ln|x-1| + \frac{13}{15}\ln|x+2| + \frac{3}{10}\left(3\arctan x + \frac{1}{2}\ln(1+x^2)\right) + C$$

or the ILT:

$$\mathcal{L}^{-1}\left\{\frac{-1/6}{s-1} + \frac{13/15}{s+2} + \frac{\frac{3}{10}(s+3)}{s^2+1}\right\}(t) = -\frac{1}{6}e^t + \frac{13}{15}e^{-2t} + \frac{3}{10}(\cos t + 3\sin t)$$

→ The first step is to obtain *proper* fractions; improper fractions, i.e., rational functions whose numerator has a degree higher or equal to the denominator will be reduced by long division, splitting off a polynomial (which can readily be integrated immediately). *In the case of ILT, this step should never be needed: Laplace transforms will go to 0 as $s \to \infty$,* *so they will always be proper fractions already. Polynomials do not have any function as an Inverse Laplace Transform. In case you learn about $\delta(x)$, $\delta'(x)$, $\delta''(x)$,... later: these don't qualify as functions, strictly speaking.*

→ Now you have to factor the denominator. This follows the general
Theorem: Any polynomial can be written as a product of linear and quadratic polynomials.
This may be difficult to carry out in *practice*, and if you get stuck here, you can't do much about the integral *(or the ILT)* either. You will have to apply this theorem to the denominator of the rational function. To do so was the first step in ∫implification (1) of our example.

Let us pause a bit here: If you want to write a quadratic polynomial, e.g., $x^2 + 4x + 3$ as a product of linear polynomials, you can always use the quadratic formula to find the zeros of that polynomial: $x^2 + 4x + 3 = 0$ if and only if $x = -3$ or $x = -1$. This is how you find $x^2 + 4x + 3 = (x + 1)(x + 3)$. If the quadratic formula does not give any real zeros, as in the case of $x^2 + 4x + 5$, you leave the quadratic polynomial alone. (Real) zeros of the polynomial will always correspond to linear factors.

In the case (1), you have no feasible systematic way to find zeros of the denominator $x^4 + x^3 - x^2 + x - 2$. By guessing, you may however find that $x = 1$ is a zero, and then you know

$$\begin{aligned} x^4 + x^3 - x^2 + x - 2 &= (x-1)(\text{poly' of deg 3, to be found by long division}) \\ &= (x-1)(x^3 + 2x^2 + x + 2) \end{aligned}$$

If you can guess another zero of the remaining factor $(x^3 + 2x^2 + x + 2)$ — here, this would be $x = -2$ —, you get $(x^3 + 2x^2 + x + 2) = (x + 2)(x^2 + 1)$

In order to see how surprisingly strong this factorization theorem is, try the polynomial x^4+1. It has no real zeros, so applying our theorem to it cannot produce linear factors. So, if our boldfaced theorem is true, it must be possible to write x^4+1 as a product of two quadratic polynomials. If you try to find how this will actually look: well, it will be quite sophisticated, you would probably not guess it. You have to find numbers p_1, q_1, p_2, q_2 such that $x^4+1 = (x^2+p_1x+q_1)(x^2+p_2x+q_2)$. *Can you do this, by expanding the right hand side and comparing coefficients of like powers of x? — In principle you can; but don't get disheartened: it takes some time to find the coefficients.* If you have actually carried it out, you'll be in for a surprising and cute formula:[1]

We can now continue our itemized strategy of finding a partial fraction decomposition:

→ **Theorem: Any proper fraction of polynomials can be decomposed into partial fractions according to the following example, which displays all features that could occur:** Given numbers $a_1, a_2, p_1, p_2, q_1, q_2$, and any polynomial in the numerator, numbers $b_1, b_2, b_3 \ldots$ can be found such that:

I'll refer to this as 'the big example'.

$$\begin{aligned} &\frac{\text{any polyn' of degree less than the degree of the denominator}}{(x-a_1)(x-a_2)^4(x^2+p_1x+q_1)(x^2+p_2x+q_2)^2} = \\ &= \frac{b_1}{(x-a_1)} + \frac{b_2}{(x-a_2)} + \frac{b_3}{(x-a_2)^2} + \frac{b_4}{(x-a_2)^3} + \frac{b_5}{(x-a_2)^4} + \\ &+ \frac{b_6x+b_7}{x^2+p_1x+q_1} + \frac{b_8x+b_9}{x^2+p_2x+q_2} + \frac{b_{10}x+b_{11}}{(x^2+p_2x+q_2)^2} \end{aligned}$$

Make sure you don't overlook the changes I suggest below for the quadratic terms. The paradigm here is correct, but NOT best possible for quadratic factors!

[1] Answer: $x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$. At least, check this by expanding.

In other words:

- For every nonrepeated linear factor in the denominator on the left (here $x - a_1$), you get one simple fraction (here, $b_1/(x - a_1)$) on the right.
- For every repeated linear factor in the denominator on the left (here $(x - a_2)^4$), you get as many simple fractions on the right as the power to which that factor was raised, and their denominators echo the corresponding factor, but with increasing powers from 1 up to the maximal power with which we started.
- For every nonrepeated quadratic factor in the denominator (here $x^2 + p_1 x + q_1$), you get one simple fraction with that very denominator. The numerator of that fraction may be a linear polynomial now, not necessarily a constant: $(b_6 x + b_7)/(x^2 + p_1 x + q_1)$.
- For every repeated quadratic factor, you get similar fractions in the PFD, each with a linear numerator, with increasing powers in the denominator.

What remains to be done is to see how yo can actually find $b_1, b_2, \ldots$. This will be described in a moment. Once you have accomplished this, you can find the antiderivative *(or the ILT)* term by term. Actually, you may not find the last term (repeated quadratic factor) routine at all, and you may not need to be able to handle this case in the practical situations encountered at your level.

This ends the basic outline of partial fraction decomposition. We'll make some improvements over this basic outline later; but first let us discuss the way how you can actually find the numbers $b_1, b_2, b_3, \ldots$.

There is a simple-minded way that always works; however, in all but the simplest cases it will be rather tedious. But you should have understood it and tried for yourself, before you venture into the more sophisticated, but very fast way of doing it. For example, assume we need to find b_1, b_2, b_3 such that

$$\frac{x^2 + 2x + 2}{x(x-1)(x+2)} = \frac{b_1}{x} + \frac{b_2}{x-1} + \frac{b_3}{x+2} \tag{2}$$

holds identically (for all x). So we bring the right hand side on a common denominator, and sort powers of x in the numerator:

$$\begin{aligned}
\frac{b_1}{x} + \frac{b_2}{x-1} + \frac{b_3}{x+2} &= \frac{b_1(x-1)(x+2)}{x(x-1)(x+2)} + \frac{b_2 x(x+2)}{x(x-1)(x+2)} + \frac{b_3 x(x-1)}{x(x-1)(x+2)} \\
&= \frac{b_1(x^2 + x - 2) + b_2(x^2 + 2x) + b_3(x^2 - x)}{x(x-1)(x+2)} \\
&= \frac{x^2(b_1 + b_2 + b_3) + x(b_1 + 2b_2 - b_3) + (-2b_1)}{x(x-1)(x+2)} \\
\text{and this should} &= \frac{x^2 + 2x + 2}{x(x-1)(x+2)}
\end{aligned}$$

So comparing coefficients in the numerator, you need

$$\begin{array}{rcl}
b_1 + b_2 + b_3 = 1 & \searrow & \\
 & \oplus \text{---} & 2b_1 + 3b_2 = 3 \\
b_1 + 2b_2 - b_3 = 2 & \nearrow & \\
-2b_1 = 2 & \text{———} & b_1 = -1
\end{array}$$

Therefore we get $b_1 = -1$, $b_2 = 5/3$, $b_3 = 1/3$.

This method is available in all cases, but it involves as many equations in as many unknowns as the degree of the denominator. Looking back to the big example on page 2, observe in the long formula, how you will automatically introduce as many unknowns $b_1, b_2, \ldots$ as the degree of the denominator. In that example, this degree was 11. In the numerator you would obtain 11 equations by comparing the coefficients of the powers $x^0, x^1, x^2, \ldots, x^{10}$. And higher powers would not occur in the numerator, because we are dealing with proper fractions. Solving 11 equations for 11 unknowns would be A LOT of work.

In contrast, here is a shorter method, which will most easily apply to linear nonrepeated factors. With a slight modification for linear repeated factors, it will only give the coefficient of the highest power (in the big example given above, it would therefore only yield b_1 and b_5). By means complex numbers in intermediate calculations, the method can also be used for quadratics in the denominator. In the big example, it would therefore also produce b_6, b_7 and b_{10}, b_{11}. So let's see how this method works for (2), where indeed we have the best possible situation: non-repeated real factors.

We determine b_1, b_2, b_3 in turn from (2), not together. To this end, we multiply (2) by the corresponding denominators respectively, namely by x, $x-1$ and $x+2$. This is done in separate independent steps starting over from (2) each time. A nice side effect of this procedure is that mistakes in the calculation of one coefficient will not affect the other coefficients, whereas in the naive coefficient comparison method one mistake poisons the whole calculation. For b_1, multiplication by x transforms (2) into

$$\frac{x^2+2x+2}{(x-1)(x+2)} = b_1 + \left(\frac{b_2}{x-1} + \frac{b_3}{x+2}\right) \cdot x \tag{3}$$

We (pretend to) plug in $x = 0$ into this equation, and get

$$\frac{0^2+2\cdot 0+2}{(0-1)(0+2)} = b_1 + (\ldots)\cdot 0$$

i.e., $-1 = b_1$ immediately. The method is called cover-up method, because it can be done without a lot of writing already from (2): To obtain b_1, you look at the left hand side, cover up exactly that term in the denominator that goes with b_1 on the right hand side, and then you plug in that number for x which would have made vanish the covered-up factor in the denominator. Similarly you would get b_2 if you multiply eqn. (2) by $(x-1)$ and after a cancellation of $(x-1)$ on the left pretend to plug in $x = 1$. (Do it, and also do it similarly to get b_3.)

I have been careful to say we *pretend to* plug in, rather than 'we plug in'. The reason is that $x = 0$ is not legitimate to plug in into (2), exactly because of the vanishing denominator. Therefore, equation (3), which was obtained from (2), is also not legitimate to be used for $x = 0$. What we actually mean to do here is to calculate the *limit* as $x \to 0$. But the actual calculation of this limit will now (i.e., after having multiplied by x) amount practically to plugging in $x = 0$.

There is kind of a philosophical message coming together with the partial fraction decomposition. You should consider as distinctive marks those points x of a rational function f where its denominator vanishes. If ever rational functions were wanted by the sheriff for wrongdoing, their vertical asymptotes would be the information given on the public announcement :-) With some embellishments added, there will be a result in advanced calculus to the effect that the behavior of a rational function near these points identifies that function nearly as uniquely as a fingerprint. (To be precise, it is a theorem from complex variables, and the fingerprinting is only possible if complex x are taken into account. Restricting oneself to real numbers is like watching a movie through a crack in the wall where you only see a small part of the screen.) To write a rational function in terms of partial fractions means to write it in such a way as to display certain of its essential features the most visibly. Displaying essential features as clearly as possible will simplify any scrutiny, in particular the search for an antiderivative *or an ILT*. And the cover up method is so smart and efficient just because it uses those numbers for x where the essential things happen, namely where the denominator of the rational function vanishes. By focusing on the essential points (in the example $x = 0$, $x = 1$ and $x = -2$) – 'essential points' in the literal as well as in the figurative sense – we avoid unnecessary calculations and retreive b_1, b_2 and b_3 exactly at those places where they naturally belong.

Now let's throw in some improvements:

(i) We want to see how to use the cover-up method with repeated factors.

(ii) We want to see a variant for quadratic factors that usually should supersede what we did in the big example.

(iii) We want to use the cover-up method with complex numbers to deal with quadratic factors as well.

(iv) We'll see a 'haphazard' method than can be very neatly used to check for miscalculations. It can also be used for calculating coefficients in a way that takes advantage of special opportunities.

(i) Repeated Factors

Example: We want to calculate the PFD

$$\frac{x^2+5x+3}{(x+2)(x-1)^3} = \frac{b_1}{x+2} + \frac{b_2}{x-1} + \frac{b_3}{(x-1)^2} + \frac{b_4}{(x-1)^3} \tag{4}$$

We get $b_1 = \frac{(-2)^2+5(-2)+3}{((-2)-1)^3} = -3/(-27) = 1/9$ by cover-up. To get b_4 by cover-up, we multiply eqn (4) with the *highest* power of $(x-1)$ that occurs, namely with $(x-1)^3$ and then take $\lim_{x\to 1}$ ('pretend to plug in $x = 1$ after cancellation'). This gives us

$$\frac{x^2+5x+3}{(x+2)} = \frac{b_1}{x+2}(x-1)^3 + b_2(x-1)^2 + b_3(x-1) + b_4$$

and letting $x \to 1$, we get $\frac{1+5+3}{1+2} = b_4$, i.e., $b_4 = 3$.

We CANNOT get b_3 by multiplying with $(x-1)^2$, because then the term $b_4/(x-1)$ remains on the rhs and prevents taking $x \to 1$. The only way to get b_3 with cover-up is to move $b_4/(x-1)^3$ to the left hand side first. b_3 is like a mouse hiding behind the elephant b_4, and you won't find the mouse unless the elephant has been moved aside:

$$\frac{x^2+5x+3}{(x+2)(x-1)^3} - \frac{3}{(x-1)^3} = \frac{b_1}{x+2} + \frac{b_2}{x-1} + \frac{b_3}{(x-1)^2} \tag{5}$$

We do this only after having found that $b_4 = 3$. The fact that we have also found b_1 already is irrelevant, and this is why I have chosen not to plug in $b_1 = 1/9$. Now you put the lhs on a common denominator, and I promise you ahead of time that a factor $x-1$ can be split off in the new numerator. Indeed

$$\frac{x^2+5x+3}{(x+2)(x-1)^3}-\frac{3}{(x-1)^3}=\frac{(x^2+5x+3)-3(x+2)}{(x+2)(x-1)^3}=\frac{x^2+2x-3}{(x+2)(x-1)^3}=\frac{(x-1)(x+3)}{(x+2)(x-1)^3}$$

and with the now re-calculated lhs, we get from (5):

$$\frac{(x+3)}{(x+2)(x-1)^2}=\frac{b_1}{x+2}+\frac{b_2}{x-1}+\frac{b_3}{(x-1)^2} \tag{6}$$

Now we multiply with $(x-1)^2$ and after cancellation we get: $\frac{(x+3)}{(x+2)}=(\ldots)(x-1)+b_3$, hence with $x\to 1$ we see $b_3=4/3$. — How did I know ahead of time that I'd get a factor $(x-1)$ in the new numerator? Well, if I couldn't have canceled this extra factor $(x-1)$ from the numerator, the rhs would have given a nice number as $x\to 1$, but the lhs would go to ∞ as $x\to 1$. So the two sides couldn't be equal! The observation that I could predict before the calculation that $(x-1)$ would factor off and cancel on the lhs has two applications: If it didn't, I would immediately detect that a miscalculation must have occurred and would look to fix it before going further. Moreover, in a more complicated example the new numerator might be a polynomial of degree 3 or more (in our example it was quadratic). Then I might have no means to find out whether I can factor the numerator, or might not see it by eyeballing. It is crucial then that I *know* ahead of time that I can pull out this factor; and I will do it by means of a long division.

Of course, to get b_2, you now move $b_3/(x-1)^2$ to the left...

(ii) A better paradigm for quadratics

Example: Suppose we have to find the PFD of $\frac{x^4+3x-10}{(x-2)(x^2+2x+4)^2}$.

According to the method laid out in the big example, you'd do the following

$$\begin{aligned}\frac{x^4+3x-10}{(x-2)(x^2+2x+4)^2}&=\frac{b_1}{(x-2)}+\frac{b_2x+b_3}{(x^2+2x+4)}+\frac{b_4x+b_5}{(x^2+2x+4)^2}\\&=\frac{b_1}{(x-2)}+\frac{b_2x+b_3}{(x+1)^2+3}+\frac{b_4x+b_5}{((x+1)^2+3)^2}\end{aligned} \tag{7}$$

Whatever you do with the PFD once you have it, be it antiderivatives *or ILT*, you will always want to complete the squares in the quadratic terms in the denominator, so this is what I have done. But now I suggest another small modification: You better try

$$\frac{x^4+3x-10}{(x-2)(x^2+2x+4)^2}=\frac{c_1}{(x-2)}+\frac{c_2(x+1)+c_3}{(x+1)^2+3}+\frac{c_4(x+1)+c_5}{((x+1)^2+3)^2} \tag{8}$$

It looks different, but in the end it amounts to the same. However, (8) rather than (7) is the form you will want for integration and ILT anyway. Moreover, it is somewhat easier for calculation, if you use the complex cover-up method I'll show you in a moment.

→ Why did I choose $(x+1)$ rather than x in the numerator for the quadratics? — Simply because $(x+1)$ is what arose in the denominator when I completed the squares. If the denominator had been $x^2+5x+7=(x+\frac{5}{2})^2+\frac{3}{4}$, I would have taken $c_2(x+\frac{5}{2})+c_3$ in the numerator.

→ Why does it amount to the same? — If you tried it like (7) and got $b_2=11/12$, $b_3=-4/12$ (as you would indeed), then I would get, from (8) that $c_2=11/12$, $c_3=-15/12$, because

$\frac{11}{12}(x+1) - \frac{15}{12} = \frac{11}{12}x - \frac{4}{12}$. We get different coefficients, but the same result, just differently written. (Of course your b_1 would be the same as my c_1.)

→ Why do I prefer (8) over (7)? — Suppose I want to integrate $\int \frac{c_2(x+1)+c_3}{(x+1)^2+3}\,dx$: Then I take it apart in the very form in which I obtained it from PFD (8), as $c_2 \int \frac{(x+1)}{(x+1)^2+3}\,dx + c_3 \int \frac{1}{(x+1)^2+3}\,dx$, and each term is a standard integral (the substitution $u = x+1$ applies). However, if I had written my PFD in the style of (7), taking it apart into $b_2 \int \frac{x}{(x+1)^2+3}\,dx + b_3 \int \frac{1}{(x+1)^2+3}\,dx$ would NOT help at all, because the first integral is not so standard: the substitution $u = x+1$ would give $\int \frac{u-1}{u^2+3}\,du$, and I would just have to take it apart again, which amounts exactly to moving from (7) to (8) anyway. This is not a big deal, but it is a small improvement. Another improvement shows up below (item (iii)), when I can calculate the c_i more easily form (8) than I could calculate the b_i from (7).

The same preference applies, when I need the PFD to do an ILT:
$\mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+3}\}(t) = e^{-t}\cos\sqrt{3}t$, *but* $\mathcal{L}^{-1}\{\frac{s}{(s+1)^2+3}\}(t)$ *needs to be taken apart in a 'non-obvious' way:*

$$\mathcal{L}^{-1}\{\frac{s}{(s+1)^2+3}\}(t) = \mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+3}\}(t) - \mathcal{L}^{-1}\{\frac{1}{(s+1)^2+3}\}(t) = e^{-t}\cos\sqrt{3}t + (1/\sqrt{3})e^{-t}\sin\sqrt{3}t\ .$$

With (8) you get the correct splitting automatically.

(iii) The complex cover-up method for quadratic terms

Complex numbers can be used to factor quadratics into two linear factors, even when this cannot be achieved with real numbers. For instance, $x^2+4x+13 = (x+2)^2+9 = (x+2+3i)(x+2-3i)$. In principle you could therefore avoid quadratics altogether and obtain a PFD with linear terms alone, e.g.,

$$\frac{x^2+x+7}{(x-1)(x^2+4x+13)} = \frac{x^2+x+7}{(x-1)((x+2)^2+9)} = \frac{x^2+x+7}{(x-1)(x+2+3i)(x+2-3i)}$$
$$= \frac{b_1}{x-1} + \frac{b_2}{x+2+3i} + \frac{b_3}{x+2-3i}$$

could do this in principle, but that's more complex variables than is convenient

We do NOT want to be so radical, because UNLESS YOU ARE GOOD WITH COMPLEX VARIABLES, AND I MEAN REALLY GOOD, YOU DON'T EVER WANT TO WRITE DOWN THE LOGARITHM OF A COMPLEX NUMBER, and logarithms is what would show up if you took the integral.

Likewise, you may shy away from $\mathcal{L}^{-1}\{1/(s+2+3i)\}(t)$, *even though it would be perfectly ok to use* $\mathcal{L}^{-1}\{1/(s+a)\}(t) = e^{-at}$ *with* $a = 2+3i$ *and use the Euler formula to get real trigs from the complex exponential.*

The basic rule about complex numbers and complex variables for users who have studied only real variables is quite simple: All the practical calculation formulas (algebra, product rule, chain rule, integration by parts) work just as well with complex numbers as with real numbers. Being ruthless und unworried wins the day. There are however a few hard hat areas, where you must not go, unless you have learned complex variables well: You do not want to have logarithms, inverse trigs or roots (non-integer powers) of any expression that takes non-real values (i.e., that involves complex numbers). Keep out of this hard-hat area and you can safely enjoy the benefits of complex numbers. The most important role of complex numbers in calculus stems from the Euler formula $e^{ix} = \cos x + i\sin x$.

> This note is only for advanced readers who do know complex variables well: Review the cover-up method and notice that you are actually calulating the residues of the poles of the rational function. You may henceforth rename the cover-up method into residue method.

So here is how you can do the PFD for our example with the complex cover-up method:

$$\frac{x^2+x+7}{(x-1)((x+2)^2+9)} = \frac{c_1}{x-1} + \frac{c_2(x+2)+c_3}{(x+2)^2+9}$$

You multiply the whole equation with the quadratic and then take the limit $x \to -2+3i$ ('pretend to plug in $-2+3i$ for x'). This is the same idea as for the linear terms, because $-2+3i$ is what makes the quadratic in the denominator vanish. (With the same rationale, you could have chosen $-2-3i$ instead of $-2+3i$; one is as good as the other.) So we get:

$$\frac{x^2+x+7}{(x-1)} = \frac{c_1}{x-1}((x+2)^2+9) + c_2(x+2) + c_3$$

and by pretending to plug in $-2+3i$ for x, we get:

$$\frac{(-2+3i)^2+(-2+3i)+7}{(-2+3i)-1} = c_2(-\not{2}+3i+\not{2}) + c_3$$

and c_1 drops out because $x=-2+3i$ makes $(x+2)^2+9$ vanish. Note also how the 2 cancels because I have followed the model (8) rather than the model (7). This will save us one division by a complex number. However, we have to suffer through the evaluation of the left hand side, there is no free lunch. We expand in such a way as to get a *real* denominator:

$$\frac{(-2+3i)^2+(-2+3i)+7}{(-2+3i)-1} = \frac{(4-12i-9-2+3i+7)}{(-3+3i)}\frac{(-3-3i)}{(-3-3i)} = \frac{(-9i)(-3-3i)}{18} = \frac{27i-27}{18}$$

So we have found that $-\frac{3}{2}+\frac{3}{2}i = c_3 + c_2(3i)$. Now this is one equation for two unknowns c_2, c_3. We know however that c_2 and c_3 must be *real* numbers. (We could have calculated them in principle with the naive method from pg. 3, and we would never have encountred complex numbers.) This is why we conclude $-\frac{3}{2} = c_3$ and $\frac{3}{2}i = c_2(3i)$. Our calculation was a bit more involved than in the case of real number cover-up, but in return it also gave us two coefficients!

If you had chosen $x=-2-3i$ instead of $x=-2+3i$, the very same calculation would have ensued, except for a single easy change, throughout: all i's would have been replaced with $-i$'s in each step. You would have gotten $-\frac{3}{2}-\frac{3}{2}i = c_3 - c_2(3i)$ instead of $-\frac{3}{2}+\frac{3}{2}i = c_3+c_2(3i)$, and this is your second equation, in case you didn't like my little shortcut with saying $c_{2,3}$ *must* be real.

In principle you can now use the 'move the elephant to the left so you can see the mouse on the right' method together with complex cover-up if you have repeated quadratic factors in the denominator. I'll forego working out an example.

(iv) Haphazard methods and error checking

Let's have another look at example (4):

$$\frac{x^2+5x+3}{(x+2)(x-1)^3} = \frac{b_1}{x+2} + \frac{b_2}{x-1} + \frac{b_3}{(x-1)^2} + \frac{b_4}{(x-1)^3}$$

We want to find b_1, b_2, b_3, b_4 from this single equation. Is it really a single equation?? Not at all, it's actually infinitely many equations!!! Because it has to be true for **every** x you choose (as long as it makes sense to plug this x in. Plug in $x=0$ and get one equation, namely $3/(-2) = b_1/2 - b_2 + b_3 - b_4$. Plug in $x=-1$ and get another equation. Choose any four x's you like or find convenient; this way you get four equations to determine the four unknowns. Your neighbor may have made different choices and gotten different equations, but his/her solutions would be the same as yours. Since you can take **any** x, it's a haphazard method. If you can choose nice numbers, you get nice equations.

We did calculate (most of) the b_i above by different methods: $b_1 = \frac{1}{9}$, $b_4 = 3$, $b_3 = \frac{4}{3}$, and if we had persevered, we would have found $b_2 = -\frac{1}{9}$. To check for mistakes we now take the equation from our haphazard choice $x = 0$, plug in the b_i we calculated and see if the equation is satisfied. The odds are that if we made a mistake, it will show up now. Or else, suppose we had found the moving over of terms tedious and had not persevered through b_2. We could then find it by using the haphazard equation from $x = 0$ (or any other, if you prefer).

The haphazard method is *not* efficient as an all-purpose tool. Rather it is an opportunitic tool that may be good for mopping up if only one coefficient is left to be calculated, or when the routine method would be tedious, and when nice numbers are available.

Note that the cover-up method is a way of choosing exactly those numbers x which seem to be illegal because they make the denominator 0. This however is exactly the wisdom of the **FBI** (**F**unction **B**ehavior **I**nvestigation). You snoop around exactly at those locations where the function is doing something bad. This way you collect the needed information most quickly. Strictly speaking, you do not really go to the illegal places, you just get close to them in the sense of a limit. This is how FBI still stays on the legal side...

One very convenient haphazard 'number' you may not think of at first is ∞. This is of course another of these limit tricks, where you just get very close to what would be illegal to plug in: Here is how it works: Multiply the equation with x, and then take the limit $x \to \infty$. In our example (4) you get 0 on the left hand side, because the numerator has degree 2 (multiply it with x, get degree 3), whereas the denominator has degree 4 (still higher than 3). On the rhs, b_3 and b_4 get killed for the same reason, and all that remains is $b_1 \lim_{x\to\infty} \frac{x}{x+2} + b_2 \lim_{x\to\infty} \frac{x}{x-1} = b_1 + b_2$. So we get the equation $b_1 + b_2 = 0$, a very easy equation indeed.

Some practice problems

Here are a few examples for practising integrals .... *(or ILT's)*: They are chosen rather on the difficult side, *and more with integration than ILT's in mind.*

(*a*) $\displaystyle\int_0^1 \frac{dx}{x^3+1}$ $\qquad$ $\displaystyle\mathcal{L}^{-1}\left\{\frac{1}{s^3+1}\right\}(t)$

(*b*) $\displaystyle\int_2^4 \frac{x^2+2x+3}{(x-1)^2(x^2+1)}\,dx$ $\qquad$ $\displaystyle\mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s-1)^2(s^2+1)}\right\}(t)$

(*c*) $\displaystyle\int_0^1 \frac{x^4+1}{x^2+1}\,dx$ $\qquad$ $\displaystyle\mathcal{L}^{-1}\left\{\frac{s^4+1}{s^2+1}\right\}(t)$

(*d*) $\displaystyle\int_0^1 \frac{dx}{x^4+1}$ this is a tough one, because of the difficulty of factoring $\qquad$ $\displaystyle\mathcal{L}^{-1}\left\{\frac{1}{s^4+1}\right\}(t)$

(*e*) $\displaystyle\int_0^1 \frac{x\,dx}{x^4+1}$ this one is much easier – simplify the integral by a substitution first! *not simpler for ILT* $\mathcal{L}^{-1}\left\{\frac{s}{s^4+1}\right\}(t)$

(*f*) $\displaystyle\int_0^1 \frac{x+1}{x^2+x+1}\,dx$ $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}(t)$

Solution to practice problems

Not all intermediate steps are carried out, but essential steps are given.

(a)

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2-x+1}$$

with $a = \frac{1}{3}$ (cover up method), $b = -\frac{1}{3}$, $c = \frac{2}{3}$ (solve linear equations for unknown coefficients; or move over $a/(x+1)$ to the other side and simplify then you can read off b, c; or else use $x = 0$ and $x \to \infty$ according to the haphazard method). Complex cover-up may be overkill here. Must complete square in denominator and separate fractions in order to reduce to standard integrals:

$$\frac{1}{3}\left(\frac{-x+2}{x^2-x+1}\right) = \frac{1}{3}\left(\frac{-x+2}{(x-\frac{1}{2})^2+\frac{3}{4}}\right) = \frac{1}{3}\left(\frac{-(x-\frac{1}{2})}{(x-\frac{1}{2})^2+\frac{3}{4}} + \frac{\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}\right)$$

The ansatz above made no use of the improvement (ii) from page 6 for the quadratics. With the improvement, the separating of fractions just done would have been obtained immediately out of the PFD.

$$\begin{aligned}\int_0^1 \frac{dx}{x^3+1} &= \frac{1}{3}\int_0^1 \left(\frac{1}{x+1} - \frac{(x-\frac{1}{2})}{(x-\frac{1}{2})^2+\frac{3}{4}} + \frac{\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}\right) dx = \\ &= \frac{1}{3}\left([\ln(x+1)]_0^1 - \left[\frac{1}{2}\ln\Big((x-\frac{1}{2})^2+\frac{3}{4}\Big)\right]_0^1 + \left[\frac{3}{2}\frac{2}{\sqrt{3}}\arctan\frac{x-\frac{1}{2}}{\sqrt{3}/2}\right]_0^1\right) \\ &= \frac{1}{3}\left(\ln 2 + 0 + \sqrt{3}\arctan(\frac{1}{\sqrt{3}}) - \arctan(-\frac{1}{\sqrt{3}})\right) = \frac{1}{3}\left(\ln 2 + \frac{\pi}{3}\sqrt{3}\right)\end{aligned}$$

Likewise, for the ILT:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^3+1}\right\}(t) &= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2+\frac{3}{4}}\right\}(t) + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-\frac{1}{2})^2+\frac{3}{4}}\right\}(t) = \\ &= \frac{1}{3}e^{-t} - \frac{1}{3}e^{t/2}\cos(\sqrt{3}\,t/2) + \frac{1}{2}\times\frac{2}{\sqrt{3}}e^{t/2}\sin(\sqrt{3}\,t/2) = \frac{1}{3}e^{-t} + \frac{1}{3}e^{t/2}(\sqrt{3}\sin(\sqrt{3}\,t/2) - \cos(\sqrt{3}\,t/2))\end{aligned}$$

(b)

$$\begin{aligned}\int_2^4 \frac{x^2+2x+3}{(x-1)^2(x^2+1)}\,dx &= \int_2^4 \left(\frac{3}{(x-1)^2} - \frac{1}{(x-1)} + \frac{x-1}{x^2+1}\right) dx = \\ &= \left[\frac{-3}{(x-1)} - \ln|x-1| + \frac{1}{2}\ln(x^2+1) - \arctan x\right]_2^4 = 2 + \arctan 2 - \arctan 4 + \frac{1}{2}\ln\frac{17}{45}\end{aligned}$$

Likewise, for the ILT:

$$\mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s-1)^2(s^2+1)}\right\}(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{s-1}{s^2+1}\right\}(t)$$
$$= (3t-1)e^t + \cos t - \sin t$$

(c)

$$\int_0^1 \frac{x^4+1}{x^2+1}\,dx = \int_0^1 \left(x^2 - 1 + \frac{2}{x^2+1}\right) dx = \frac{1}{3} - 1 + 2\arctan 1 = \frac{\pi}{2} - \frac{2}{3}$$

For ILT this was a trick problem, because there is no such function whose Laplace transform is $(s^4+1)/(s^2+1)$: every function's Laplace transform has limit 0 as $s \to 0$, but $(s^4+1)/(s^2+1)$ doesn't. If you forgot this, you would have run into trouble when s^2 couldn't be found in the table of ILT's. If at some time you have learned about pulses like $\delta(\cdot)$ and their derivatives, which strictly speaking are not functions, then you can do the ILT and it is not a trick question any more.

(d)

Factorization of denominator: see footnote and text on page 2. If you know that $e^{i\pi} = -1$, you can see that $x^4+1=0$ has the solutions $x = e^{i\pi/4} = \cos\pi/4 + \sin\pi/4 = (1+i)\sqrt{2}/2$, $x = e^{-i\pi/4} = \ldots$, $x = e^{3i\pi/4} = \ldots$, and $x = e^{-3i\pi/4} = \ldots$. Pairing complex conjugate ones

$$x^4+1 = \Big((x-e^{i\pi/4})(x-e^{-i\pi/4})\Big) \times \Big((x-e^{3i\pi/4})(x-e^{-3i\pi/4})\Big)$$

into quadratics that will have real coefficients after evaluation is another way of obtaining the factorization from page 2.

$$\frac{1}{x^4+1} = \frac{ax+b}{x^2+\sqrt{2}x+1} + \frac{cx+d}{x^2-\sqrt{2}x+1}$$

The amendment (ii) from page 6 has been neglected here, out of a fluke. So we'll have to rewrite the numerators after finding the coefficients.

With real-variable tools alone, there is probably no shortcut to solving the equations for a, b, c, d:

$$\begin{array}{ll} x^0: & b+d=1 \\ x^1: & a+c+(d-b)\sqrt{2}=0 \\ x^2: & b+d+(c-a)\sqrt{2}=0 \\ x^3: & a+c=0 \end{array} \qquad \begin{array}{l} \text{“}x^3\text{”\&“}x^1\text{”}: \quad d=b \\ \Longrightarrow \text{(with “}x^0\text{”)} \quad b=d=\tfrac{1}{2} \\ \Longrightarrow a=\sqrt{2}/4=-c \end{array}$$

Alternatively, complex cover-up can be considered to obtain the PFD. Note also that the haphazard choices $x=0$ and $x\to\infty$ yield the convenient equations $b+d=1$, $a+c=0$ very quickly.

$$\int_0^1 \frac{(\sqrt{2}/4)x+1/2}{x^2+\sqrt{2}x+1}\,dx = \frac{\sqrt{2}}{4}\int_0^1 \left(\frac{x+\sqrt{2}/2}{(x+\sqrt{2}/2)^2+1/2} + \frac{\sqrt{2}/2}{(x+\sqrt{2}/2)^2+1/2}\right) dx =$$
$$= \frac{\sqrt{2}}{8}\left[\ln\Big((x+\sqrt{2}/2)^2+1/2\Big)\right]_0^1 + \frac{\sqrt{2}}{4}\left[\arctan\frac{x+\sqrt{2}/2}{\sqrt{2}/2}\right]_0^1 =$$
$$= \frac{\sqrt{2}}{8}\ln(2+\sqrt{2}) + \frac{\sqrt{2}}{4}\Big(\arctan(1+\sqrt{2}) - \frac{\pi}{4}\Big)$$

Similarly,

$$\int_0^1 \frac{(-\sqrt{2}/4)x + 1/2}{x^2 - \sqrt{2}x + 1}\,dx = -\frac{\sqrt{2}}{4}\int_0^1 \left(\frac{x - \sqrt{2}/2}{(x-\sqrt{2}/2)^2 + 1/2} - \frac{\sqrt{2}/2}{(x-\sqrt{2}/2)^2+1/2}\right) dx =$$
$$= -\frac{\sqrt{2}}{8}\Big[\ln\Big((x-\sqrt{2}/2)^2 + 1/2\Big)\Big]_0^1 + \frac{\sqrt{2}}{4}\left[\arctan\frac{x-\sqrt{2}/2}{\sqrt{2}/2}\right]_0^1 =$$
$$= -\frac{\sqrt{2}}{8}\ln(2-\sqrt{2}) + \frac{\sqrt{2}}{4}\Big(\arctan(-1+\sqrt{2}) + \frac{\pi}{4}\Big)$$

Now $\ln(2+\sqrt{2}) - \ln(2-\sqrt{2}) = \ln\frac{(2+\sqrt{2})}{(2-\sqrt{2})} = \ln\frac{\sqrt{2}+1}{\sqrt{2}-1} = \ln((\sqrt{2}+1)^2)$. You are certainly not expected to know that $\arctan(\sqrt{2}+1) = 3\pi/8$, $\arctan(\sqrt{2}-1) = \pi/8$, but these are true, and so the final result can be simplified to

$$\int_0^1 \frac{dx}{x^4+1} = \frac{\sqrt{2}}{4}\Big\{\ln(\sqrt{2}+1) + \pi/2\Big\}$$

Likewise for the ILT:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4+1}\right\}(t) = \frac{\sqrt{2}}{4}\mathcal{L}^{-1}\left\{\frac{(s+\sqrt{2}/2)+(\sqrt{2}/2)}{(s+\sqrt{2}/2)^2+(\sqrt{2}/2)^2}\right\}(t) - \frac{\sqrt{2}}{4}\mathcal{L}^{-1}\left\{\frac{(s-\sqrt{2}/2)-(\sqrt{2}/2)}{(s-\sqrt{2}/2)^2+(\sqrt{2}/2)^2}\right\}(t)$$
$$= \frac{\sqrt{2}}{4}\left(e^{-\sqrt{2}t/2}\left(\cos\tfrac{\sqrt{2}}{2}t + \sin\tfrac{\sqrt{2}}{2}t\right) - e^{\sqrt{2}t/2}\left(\cos\tfrac{\sqrt{2}}{2}t - \sin\tfrac{\sqrt{2}}{2}t\right)\right)$$

Since in this particular example, there is a simple pattern in terms of complex numbers (namely the four zeros of the denominator mentioned above), which in the real PFD produces a pattern that is more dazzling than simple, the very best option in this example is to go complex all the way, as was considered but not pursued on pg. 7. Namely write

$$\frac{1}{s^4+1} = \frac{a}{s-e^{i\pi/4}} + \frac{b}{s-e^{-i\pi/4}} + \frac{c}{s-e^{3i\pi/4}} + \frac{d}{s-e^{-3i\pi/4}}$$

and get a, b, c, d by the very simplest version of cover-up (but with complex numbers), and l'Hôpital rather than algebra: $a = \lim_{s\to e^{i\pi/4}}(s-e^{i\pi/4})/(s^4+1) = \lim_{s\to e^{i\pi/4}} 1/4s^3 = \frac{1}{4}e^{-3i\pi/4}$, and similarly $b = \frac{1}{4}e^{3i\pi/4}$, $c = \frac{1}{4}e^{9i\pi/4} = \frac{1}{4}e^{i\pi/4}$, $d = \frac{1}{4}e^{-i\pi/4}$. The ILT will involve only exponentials, which you convert to real trigs using Euler's formula. While this method only uses ingredients available to you by the time you do ILT's and Euler's formula, the wisdom to choose this particular route of calculation would usually only be gained by a deeper knowledge of complex variables. So you shouldn't feel you yourself ought to have done it this way.

I leave the rest of the ILT's for you to figure out and stick with the integrals only. Note that in (e) we have no shortcut for ILT's, because there is no substitution rule for ILT's.

(e)

$$\text{subst. } u = x^2\text{:}\qquad \int_0^1 \frac{x\,dx}{x^4+1} = \left[\frac{1}{2}\arctan(x^2)\right]_0^1 = \frac{\arctan 1 - \arctan 0}{2} = \frac{\pi}{8}$$

(f)

$$\int_0^1 \frac{x+1}{x^2+x+1}\,dx = \int_0^1 \left(\frac{x+1/2}{(x+1/2)^2+3/4} + \frac{1/2}{(x+1/2)^2+3/4}\right) dx = \frac{1}{2}\ln 3 + \frac{\pi}{6\sqrt{3}}$$