**UTK – M231 – Differential Equations – Jochen Denzler**
**Notes on Chapter 1 of Nagle/Saff/Snider**

## Sec 1.1: Background

A key point is that differential equations ask to find an unknown *function* (rather than an unknown number as in high school equations), from a relation connecting that function with its derivative(s). If the function is a function of a single variable (often this variable will be time), we call the differential equation an **Ordinary Differential Equation (ODE)**; if the unknown function is a function of several variables (often these will be the space coordinates and, possibly time), the derivatives occurring in the differential equation will necessarily be partial derivatives, and therefore the differential equation will be called a **Partial Differential Equation (PDE)**. – Except for pointing out this basic distinction, this course will restrict attention to ODEs.

The highest order of a derivative occurring in a DE is called the order of the differential equation. The significance of this concept is as follows: As you solve the equation, you (hopefully) take a calculational procedure which, at its end, has stripped the function of all its derivatives, by means of integrations. Each integration kills one derivative and introduces a constant of integration. So the DE will not have merely a single solution, but a family of solutions, with as many arbitrary constants as is the order of the DE. — In practice, such a calculational procedure may not be available, and if it is, it's not obvious how to find it, but the conclusion concerning the number of arbitrary constants still holds true. Two example calculations are on p. 2 of the book. — Blatant exceptions to the rule "as many constants as the order of the ODE" are due to reasons more trivial than ODEs. (Eg, $\cos(y') = 5$ has no real solutions, but you don't need to know anything about ODEs to understand why.)

The DE is the mathematical representation of a law of nature (eg., free fall, radioactive decay), the free constants in the solution represent the variety of particular situations governed by these laws (e.g., free fall from what initial height; or initial amount of radioactive material).

Linear vs. nonlinear equation: be sure to note that the word "linear" refers to the *unknown function and its derivatives* (i.e., the *dependent* variable), not to the quantity of which it is a function (i.e., not to the independent variable): $y'(x) + y(x) = \sin x$ is a linear equation, in spite of the $\sin x$, because $y$ is the unknown function, so $x$ has no vote in deciding linear vs. nonlinear. In contrast, $y'(x)y(x) = x$ is nonlinear because $y$ and $y'$ (the unknown function and its derivative) get multiplied with each other.

Looking back from later in the semester (Ch. 4), you will see the significance of this definition: Linearity is what gives you the powerful superposition principle, and that superposition principle makes linear equations so much more manageable than nonlinear ones. This remark applies to all kinds of equations: systems of linear equations (for numbers), ODEs, PDEs, and others you haven't met yet and needn't know here: difference equations, integral equations.

Note: The names for the variables will change, depending on the application from which the ODE comes. If the independent variable has the meaning of a time, we will call it $t$. In other contexts we will call the independent variable $x$, following the Calc'1 customs. — If you think this switching of symbols is a nuisance, you might also be among those who consider that whole Math thing a drudgery. In that case, be alerted that the nuisance is to your benefit, eventually. Looking at symbols not by their name but by the role they play, whatever their name, forces you to think conceptually rather than in terms of meaningless abstract symbols, which is the only way to make Math not a drudgery, but something intelligible.

## Sec 1.2: Solutions and Initial Value Problems

re: implicit solution

The example and definition of "implicit solution" in the book may seem to come out of thin air: Here's more like an entire story concerning it:

Suppose you need to solve $y'(x)(1+y(x)^2) = 2x$ subject to the condition $y(1) = 3$. A stroke of ingenuitiy (or a real good calculus mastery) makes you recognize the left hand side as a total derivative:

$$\begin{aligned} y'(x)(1+y(x)^2) &= 2x && \text{which can be written as} \\ \frac{d}{dx}\left(\tfrac{1}{3}y(x)^3 + y(x)\right) &= \frac{d}{dx}x^2 && \text{and now integrate:} \\ \tfrac{1}{3}y(x)^3 + y(x) &= x^2 + C \end{aligned}$$

and because $y(1) = 3$, you conclude $\frac{1}{3}3^3 + 3 = 1^2 + C$, hence $C = 11$. By now, the derivatives are gone, and ODEwise, your job is finished; the solution is $\frac{1}{3}y^3 + y = x^2 + 11$; but one would want to do some algebra and solve for $y = y(x)$, the unknown function; and you cannot do this algebra job. So you call what you have great progress and "an implicit solution".

Actually, we should at least be sure that there is indeed a function $y(x)$ satisfying the algebraic equation $\frac{1}{3}y^3+y = x^2+11$. (For a contrast, there is no function satisfying the algebraic equation $y^2+y = -x^2-1$, and you may wish to pause a moment to think why.) So it may be a bit premature to speak of an implicit "solution" at this stage. But we'll see in a moment that ODEs do have solutions under very general asumptions, therefore those algebraic equations we arrive at when coming from ODEs will not be a dead end.

Note a slight abuse of language: A solution is a solution. Explicit vs. implicit does *not* distinguish different *kinds* of solution, but different amounts of *our knowledge* about the solution. Think of "explicit/implicit solution" as a shorthand for "solution given explicitly/implicitly".

re: Existence and uniqueness for initial value problems

An **Initial Value Problem (IVP)** for a 1st order ODE has the form

$$y'(t) = f(t, y(t)) \,, \quad y(t_0) = y_0 \tag{$*$}$$

with $f$, $t_0$, $y_0$ given, and with the task being to find a function $y = y(t)$ satisfying these two equations (the ODE and the so-called initial condition $y(t_0) = y_0$). The sought-for function has to be defined *in some interval containing* $t_0$. We expect IVPs to have *one and only one* solution. A theorem (below) proves this expectation to be right in many cases, but there are exceptions where our expectation is wrong.

The intuition behind the expectation is as follows: We are given $y$ at "time" $t = t_0$, and then, the ODE tells us also the rate of change $y'(t_0)$ at that time, so we have a good idea what $y$ is at a short time later, say at time $t_1 = t_0+\Delta t$: Namely it should be $y(t_1) = y(t_0+\Delta t) \approx y(t_0)+y'(t_0)\Delta t = y(t_0)+f(t_0, y(t_0))\Delta t$. From there on, we do another time step by the same argument, arriving at $t_2 = t_1 + \Delta t$, etc. (This is actually the simplest method of solving an IVP numerically.)

**Thm:** (Picard-Lindelöf existence and uniqueness theorem) If $f$ and $\partial f/\partial y$ are continuous in a neighborhood of $(t_0, y_0))$, then the IVP $(*)$ has one and only one solution in some (possibly smaller) neighborhood of $t_0$. (Refer to the lecture for an intuitive mening of the word neighborhood.)

**Explanations:** Take the example $y' = 1 + y^2$, $y(0) = 0$. The theorem applies because, in this example we have $f(t, y) = 1 + y^2$ and $\frac{\partial f}{\partial y}(t, y) = 2y$, and both are continuous for $(t, y)$ close to $(t_0, y_0) = (0, 0)$ (actually for all $(t, y)$). So we know there is a solution; the thm doesn't tell us how to find it, but you can check by plugging in that $y(t) = \tan t$ is a solution to this IVP. The theorem now tells us that it is the only solution. If you plot this solution, you see that $y(t) \to +\infty$ as $t \to \frac{\pi}{2}-$, and $y(t) \to -\infty$ as $t \to -\frac{\pi}{2}+$.

Even though the function $y(t) = \tan t$, *as a formula*, is defined outside the open interval $t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, *as a solution to the IVP*, we should think of it as defined *only* in this interval, because it cannot be continuously differentiable in any larger *interval*, only in a larger *set* that is not an interval any more. In DEs that model an explosion, this behavior is to be expected (and it is generally named "blow up in finite time" based on this idea). The ODE $y' = 1 + y^2$ shows nothing suspicious for $t = \frac{\pi}{2}$ nor for any particular $y$. So you see why I stressed the "possibly smaller neighborhood" in the theorem.

**Thm:** *Linear* ODEs *never* show the "blow up in finite time" phenomenon, nonlinear ODEs may or may not have it (depending on the particular ODE or IVP).

**Thm:** (Peano existence theorem) If $f$ (but not necessarily $\partial f/\partial y$) is continuous in a neighborhood of $(t_0, y_0))$, the IVP still has a solution, but it may have more than one. — The exercises give an example.

**Rmk:** These three theorems hold in more generality, namely for the higher order IVP

$$y^{(n)}(t) = f(t, y(t), y'(t), \ldots, y^{(n-1)}(t)) \,, \quad y(t_0) = y_0 \,,\; y'(t_0) = y_1 \,,\; \ldots \,,\; y^{(n-1)}(t_0) = y_{n-1}$$

(Picard-Lindelöf then assumes continuity of $f$ and all its first partial derivatives except $\partial f/\partial t$, Peano still only continuity of $f$). The theorems also hold for systems of ODEs, like, e.g.,

$$\begin{aligned} y'(t) &= f(t, y(t), z(t)) \,, \quad y(t_0) = y_0 \\ z'(t) &= g(t, y(t), z(t)) \,, \quad z(t_0) = z_0 \end{aligned}$$

(the respective assumptions must then be made for $f$ and $g$). The hypothesis for Picard-Lindelöf can be slightly weakened, but the stated version is good enough for most purposes.

## Sec 1.3: Direction Fields

Not much comment needed on this section. Only one, concerning the figure on p. 19, and the text accompanying it on p. 18: They give a direction field for $dy/dx = 3y^{2/3}$, where the IVP has *several* solutions, and say the direction field is "intriguing" for that very reason. However, if you try to look at the direction field *alone*, not seeing the ODE actually given as an equation, you may not see much difference to, e.g., the direction field for $dy/dx = |y|$, for which uniqueness does hold (even though our version of the existence and uniqueness theorem doesn't tell us so).

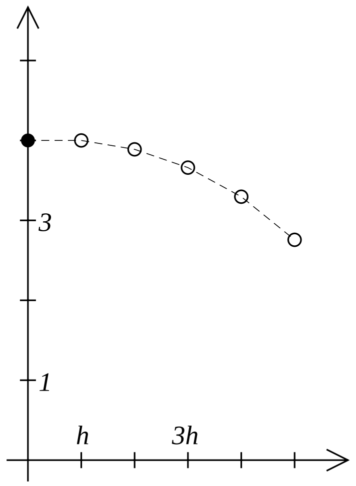
If you mark on the $y$-axis all those values of $y$ for which they have actually drawn slopes in the figure, these are only finitely many values (seven positive, seven negative, and 0). From only fifteen (or any finite number of) values of $f(y) = 3y^{2/3}$, you cannot determine that $f(y) = 3y^{2/3}$ is not differentiable at 0; for this you need all values in some neighbourhood of 0. So don't expect to *see* the intriguing thing in the direction field (left figure);

## Sec 1.4: Euler's Approximation Method

We just touch this briefly: less for its own sake, but because I think it deepens your understanding of the preceding material, in particular the Picard Lindelöf result.

A cheap technique for finding approximate solutions for an IVP $y' = f(x,y)$, $y(x_0) = y_0$, from the direction field is as follows: Locate the initial point $(x_0, y_0)$ calculate the slope $y'(x_0) = f(x_0, y_0)$ directly from the ODE, follow a short straight line segment passing over an $x$-interval of length $h$, where $h$ is a small number you choose at your convenience. The segment is approximately a part of the graph of the actual solution, and it abuts in a pint $(x_1, y_1)$ with $x_1 = x_0 + h$, $y_1 = y_0 + f(x_0, y_0)h$. Now you repeat the process with starting point $(x_1, y_1)$, etc, until you have constructed enough of the graph.

Let's denote by $y_{[h]}$ the function whose graph is the polygon of line segments and take a problem from the book, namely $dy/dx = -x/y$, $y(0) = 4$ as an example.




Here is a **Homework**: For $h = 0.1$ and $h = 0.05$, calculate $y_{[h]}(x)$ at $x \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$. (If you have a programmable pocket calculator and get the hang of it, you may also try it with $h = 0.02$, but otherwise the former two are good enough.)
This is a test example, where, unlike more realistic examples, a solution in terms of an explicit formula can be found (and you'll soon learn how): it is $y = \sqrt{16 - x^2}$, and you can of course check easily that this is a solution. Compare the above found values of the approximate solutions $y_{[h]}$ with the corresponding values of the exact solution $y$.

| $x$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|
| $y_{[0.1]}$ | 4 | | | | | | | | |
| $y_{[0.05]}$ | 4 | | | | | | | | |
| $y$ | 4 | | | | | | | | |

From this example you should first of all understand that it is plausible to expect: every 'decent' 1st order IVP (ODE plus initial value given) should have precisely one solution, no more, no less. The Picard Lindelöf theorem mentioned a few pages ago confirms this reasonable expectation for ODEs in which $y'$ is given as a continuously differentiable expression in $y$ and $x$; in contrast if $y'$ is given as an expression no better than continuous, there may be several solutions, contrary to our plausible expectation.

Next note that this method is, in principle, a way to find approximate solutions to all IVPs numerically, with no special wisdom on ODEs needed. In the rest of the semester, we'll rather learn precise, formula methods for a small selection of the most useful ODEs.

In practice, if you do treat ODEs numerically, more sophisticated numerical methods than the one outlined here are appropriate; but this one here is the starting point for understanding them all, and it's also useful to deepen the understanding of what we've learned so far.

Solution of Hwk from previous page:

We calculate $y_{[h]}(x)$ successively at $x = h$, $x = 2h$, $x = 3h$, ..., and we record the values needed for the table, when we encounter them.

We have $y_{[h]}(0) = 4$, the initial value, and then, as described in the lecture

$$y_{[h]}((n+1)h) = y_{[h]}(nh) + h \times (-nh)/y_{[h]}(nh)$$

for $n = 0, 1, 2, \ldots$ successively. Here are the results, with 5 digits behind the decimal point:

| $x$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|
| $y_{[0.1]}$ | 4 | 4. | 3.9975 | 3.9925 | 3.98498 | 3.97495 | 3.96237 | 3.94722 | 3.92949 |
| $y_{[0.05]}$ | 4 | 3.99938 | 3.99625 | 3.99062 | 3.98247 | 3.97179 | 3.95856 | 3.94276 | 3.92435 |
| $y_{[0.02]}$ | 4 | 3.999 | 3.9955 | 3.98949 | 3.98096 | 3.96989 | 3.95627 | 3.94007 | 3.92125 |
| $y$ | 4 | 3.99875 | 3.995 | 3.98873 | 3.97995 | 3.96863 | 3.95474 | 3.93827 | 3.91918 |