Introduction: Outline of MVC (Multi-Variable Calculus)

This is a calculus for functions of several (two, three, ...) variables as opposed to single-variable calculus you've seen in the first two semesters.

First I want to convince you that you have known multivariable functions since grade school! Example: plus $\quad$ plus $(5,3) = 8$

times $\quad$ times $(7,4) = 28$

Of course at that time, you did not use function notation, but rather wrote eg $5+3$ or $7 \cdot 4$ respectively.

You can build "complicated" MV functions with formulas, like eg.

$$f(x,y,z) = (\ln z + e^x \cos y) / (1+x^2+y^2)$$

or express them by "meaning" even in the absence of formulas, eg.

let $T(x,y,z,t)$ be ~~temp~~ the temperature T as a fct of location (x,y,z) and time t.

We will want to study derivatives (rates of change) of such MV fcts,

eg. $\frac{\partial T}{\partial x}$ = rate of change of temp. as location varies in x-direction

$\frac{\partial T}{\partial y}$ = —"— y

(When we do this in detail, I'll explain why we use "curly d's" ∂ instead of the usual d's you know from SVC)

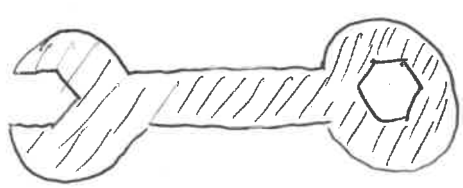
Later we will study integrals of MV fcts, and they come in quite a variety of kinds.

Ultimately, we'll see several analogs of $\int_a^b f'(x)\,dx = f(b) - f(a)$, and they are highly relevant in physics & engineering, e.g in electromagnetism.

(End of the preview)

Let's establish some <u>language</u>: the "stage" on which MV Calc plays out.

- Instead of the set $\mathbb{R}$ consisting of real numbers (usually called x) and represented geometrically as a <u>line</u>, we may have
- the set $\mathbb{R}^2$ consisting of pairs of real numbers (x,y). $\mathbb{R}^2$ is geom. represented as a <u>plane</u>. Or,
- the set $\mathbb{R}^3$ consisting of triplets of real numbers (x,y,z). $\mathbb{R}^3$ is geom. repres'd as <u>space</u>
- the set $\mathbb{R}^4$ consisting of 4-tupels (x,y,z,t). It needs fantasy to imagine a geom. representation for $\mathbb{R}^4$. But just like a blind chess player who does not <u>see</u> the knight move from C8 to D6, we can happily work with (x,y,z,t) without <u>seeing</u> the 4-dim. "chess board" on which this point is located.
- Or we may only need <u>subdomains</u> (think "portions") of eg $\mathbb{R}^2$, like

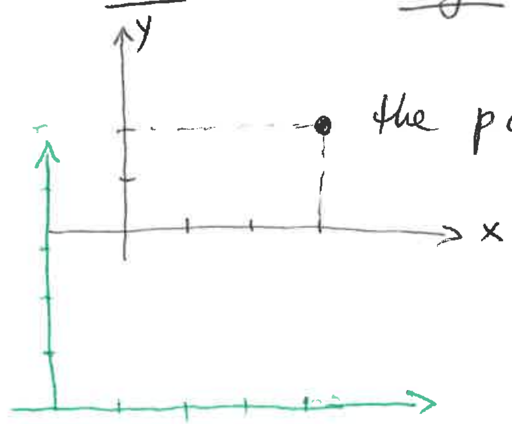
 $\Omega \subset \mathbb{R}^2$

eg if our function means to represent <u>magnetization</u> of this socket wrench ...

Points in the plane are represented by pairs of real numbers (x, y) often called coordinates.

Likewise points in space are rep'd by triplets (x, y, z).

To achieve this representation, we need a coordinate system with axes and an origin, often a cartesian coord. system



same direction of axes, but different origin would assign different numbers to the geometrically same point eg $(4.2, 5)$

And of course, with different directions of axes you'd also get different coordinates for the same geometric point.

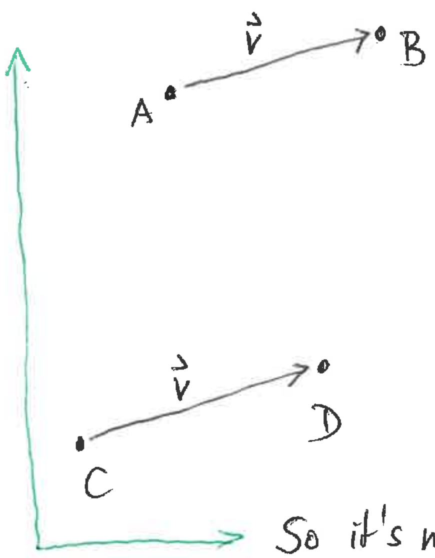
(In many cases, there is no a-priori geometry, eg if you consider pressure p and temperature T of some gas and just put them on ad-hoc axes

there is no ~~point~~ sense in rotating axes at all!)

A different critter are <u>vectors</u>. The word comes from a Latin verb that means "drive" (as in drive a ~~car~~ vehicle)

here you have that same Latin word hidden again

Vectors represent a transport "from A to B", more precisely a net effect of transport (<u>direction</u> & <u>distance</u>)



A vector $\vec{v}$

(In handwriting, I may stress that $\vec{v}$ is a vector by putting an arrow on top of the letter. In print, they prefer boldface for vectors)

Here is an equivalent version of this vector:
same distance & same direction $\rightarrow$ "same" vector

So it's not the start & endpoint that makes the vector, but rather the <u>difference</u> btw end and start points

Let me add a coordinate system: We may have

$C = (1,2) \quad D = (7,4)$ $\qquad \vec{v} = \langle 7-1, 4-2\rangle$ or

$A = (3,9) \quad B = (9,11)$ $\qquad \vec{v} = \langle 9-3, 11-9\rangle$

either way, $\vec{v} = \langle 6,2\rangle$

Vectors may be translated freely.

If you keep the <u>directions</u> of the axes, but <u>move</u> the <u>origin</u> as done on the previous page, vector coordinates don't change, whereas point coordinates do! That's why they should be distinguished notationally. The book uses angle brackets, I will frequently use square brackets with a little ^{T}: $[6,2]^T$ for $\langle 6,2\rangle$.

The reason for the notation $[6,2]^T$, or as a column $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ rather than the textbook's $\langle 6, 2 \rangle$ lies in the interaction of vectors with matrices, as discussed in 251 Matrix Algebra. If you haven't seen this material yet, simply consider $[6,2]^T$ or $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ as an alternative notation

<u>In space</u>, vectors have three components:

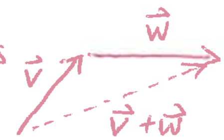
$$[u, v, w]^T \quad \text{or} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad \text{or the textbook's} \quad \langle u, v, w \rangle.$$

<u>Note:</u> Components of vectors "ought to be denoted" $[\Delta x, \Delta y, \Delta z]^T$ rather than $[x, y, z]^T$ or $[u, v, w]^T$ because they are differences of point coordinates. But we frequently don't care and disobey this "ought to", just for simplicity of notation.

Another wicked ploy to make life confusing is to blur the distinction between points and vectors, by <u>choosing an origin</u> and confusing the vector from the origin $(0,0,0)$ to the point (x,y,z), namely the vector $[x,y,z]^T$ with the point (x,y,z) itself.

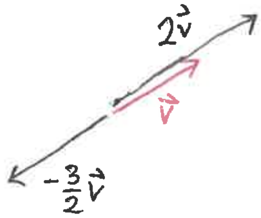
Now, for what we can do with vectors, refer to Sec 12.1 and 12.2 of the textbook, no need for me to copy this material.

You learn how you can add vectors



geometrically, or algebraically by adding their components: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

and also how to multiply vectors with numbers:



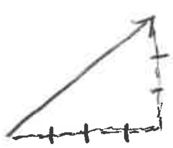
geometrically, or algebraically by multiplying the components with numbers:

$$\vec{v} = \begin{bmatrix} 1.2 \\ 1 \end{bmatrix}, \quad 2\vec{v} = \begin{bmatrix} 2.4 \\ 2 \end{bmatrix},$$

$$-\frac{3}{2}\vec{v} = \begin{bmatrix} -1.8 \\ -1.5 \end{bmatrix}$$

Numbers, whenever they hang around with vectors, like to be called <u>scalars</u>. Quirky habit, but nothing to be afraid of :-)

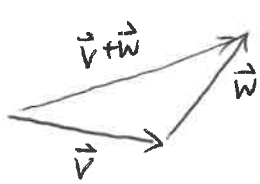
The <u>length</u>, or <u>magnitude</u> of a vector is also called the <u>norm</u> of this vector, and is denoted by $\|\cdot\|$. In cartesian coord's, you can use Pythagoras to calculate the norm



$$\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad \|\vec{v}\| = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5$$

The <u>triangle</u> inequality $\quad \|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$

represents the fact that in a triangle, one side cannot be longer than the sum of the two others:



All these notions carry over seamlessly from vectors in $\mathbb{R}^2$ to vectors in $\mathbb{R}^3$