Notes sketchy! Refer to 12.4 in book, too

Cross Product

only in $\mathbb{R}^3$

$\vec{vec} \times \vec{vec} = \vec{vec}$

Def in coordinates (right-handed cartesian)

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} := \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}.$$

$$\begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix}$$

"a 2×2 determinant"

$(\vec{u} \times \vec{v}) \cdot \vec{u} =$ 6 terms which cancel ea/other out $= \vec{0}$

same calc' for $(\vec{u} \times \vec{v}) \cdot \vec{v}$

$$\|\vec{u} \times \vec{v}\|^2 = (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2$$

$$\|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2$$

eval' each of these & compare

and yeah! they're equal!

$$\boxed{\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2}$$

$$= \|\vec{u}\|^2 \|\vec{v}\|^2 \big(\underbrace{1 - \cos^2 \measuredangle(\vec{u}, \vec{v})}_{\sin^2 \measuredangle(\vec{u}, \vec{v})}\big)$$

by taking $\sqrt{}$ and observing

$\sin \measuredangle$ is ≥ 0 for $\measuredangle$ b/w $0°$ and $180°$

find

$$\boxed{\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|v\| \sin \measuredangle(\vec{u}, \vec{v})} \quad (*)$$

this info, plus the fact that

$\vec{u} \times \vec{v} \perp \vec{u}, \vec{v}$

leaves only two options (one is $-$(the other))

More detailed pondering (skipped here) tells:

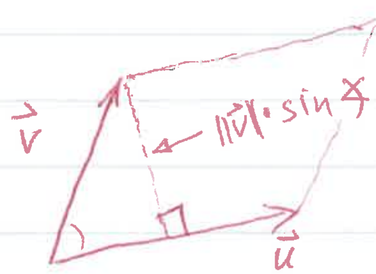
$\vec{u}, \vec{v}, \vec{u} \times \vec{v}$ (in this order)

follow the right hand rule

4 fingers of right hand point how to turn from $\vec{u}$ to $\vec{v}$ the shortest route;

thumb in hitchhike position gives direction of $\vec{u} \times \vec{v}$

Back to $(*)$:



area $= \|\vec{u}\| \cdot \|\vec{v}\| \sin \measuredangle = \|\vec{u} \times \vec{v}\|$

tricking the × product into working for vectors in the plane

consider $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ as

$$\begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \in \mathbb{R}^3$$ ditto for $\vec{v}$

$$\begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

<u>Note:</u> $\vec{u} \times \vec{v} = - \vec{v} \times \vec{u}$

Also paren's do matter

$\vec{u} \times \vec{v} \times \vec{w}$ is MEANINGLESS!

$$(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$$