Recall: a vector field in a domain $D \subset \mathbb{R}^3$ is a function $\vec{F}$ assigning to each point $\vec{x} \in D$ a vector $\vec{F}(\vec{x}) \in \mathbb{R}^3$

Similarly we have vector fields in $D \subset \mathbb{R}^2$, where the vectors $\vec{F}(\vec{x})$ are also 2-component vectors.

Examples: Force field (or acceleration field experienced by particle due to gravity)

Electric field

Magnetic field

Velocity field of a fluid $<$ gas / liquid (think of them as stationary, i.e. time-independent)

How can we draw / graph such a vector field?

Easy: at "each" point x, attach the vector $\vec{F}(x)$

↳ representative sample

For "nice" graphs, we'd like to avoid that the drawn vectors cross each other, but there is nothing that would guarantee that we get that wish

But there are a few choices we can make to help getting nice pictures:

(1) don't sample too many points

(2) graph $k\vec{F}$ instead of $\vec{F}$ (with k small)

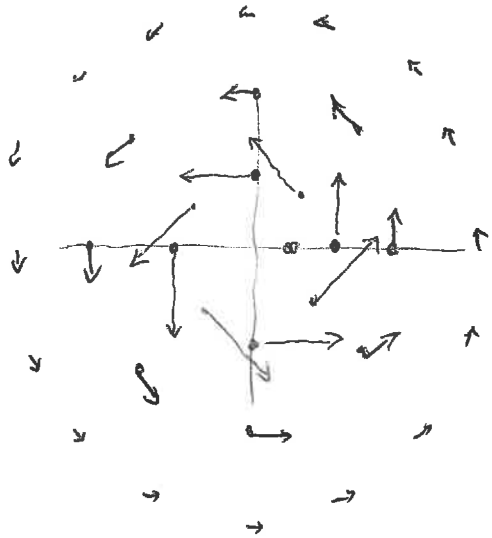
This is tantamount to using different scales for points and vectors

Examples: $\vec{F}(x,y) = \begin{bmatrix} -y/(x^2+y^2) \\ x/(x^2+y^2) \end{bmatrix}$

Notice at point (x,y) with $r = \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|$

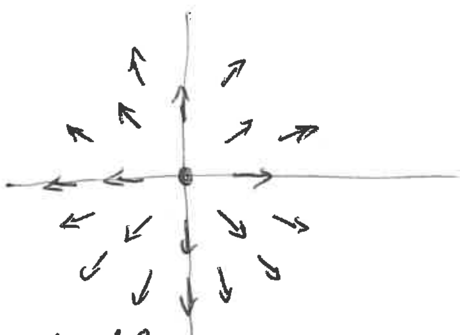
$\|\vec{F}(x,y)\| = \frac{1}{r}$

also $\vec{F}(x,y) \perp \begin{bmatrix} x \\ y \end{bmatrix}$



Ex $\vec{F}(x,y) = \begin{bmatrix} x/\sqrt{x^2+y^2} \\ y/\sqrt{x^2+y^2} \end{bmatrix}$ $\qquad \|\vec{F}\| = 1 \qquad \vec{F} \parallel \begin{bmatrix} x \\ y \end{bmatrix}$

in other words $\vec{F}(\vec{x}) = \dfrac{\vec{x}}{\|\vec{x}\|}$ $\qquad \vec{x} \neq \vec{0}$



geometrically meaningful
✓ Differentiation operations on vector fields

$$\text{curl}\,\vec{F} = \begin{bmatrix} \partial F_3/\partial y - \partial F_2/\partial z \\ \partial F_1/\partial z - \partial F_3/\partial x \\ \partial F_2/\partial x - \partial F_1/\partial y \end{bmatrix}$$

(defined wrt coord's (x,y,z) but I claim there is a coord-independent meaning behind it)

But bear with me to explain

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

divergence

(—"—)

$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F}$ (BTW sometimes they write $\operatorname{rot} \vec{F}$ for $\operatorname{curl} \vec{F}$)

(rotation)

$\operatorname{div} \vec{F} = \vec{\nabla} \bullet \vec{F}$

A few neat formulas:

$$\operatorname{curl} \vec{\nabla} f = \vec{\nabla} \times \vec{\nabla} f = \vec{0}$$

$$\operatorname{div} \operatorname{curl} \vec{F} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$$

$$\operatorname{div} \vec{\nabla} f = \frac{\partial}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial}{\partial y} \frac{\partial f}{\partial y} + \frac{\partial}{\partial z} \frac{\partial f}{\partial z} =$$ sum of the nonmixed 2nd partial deriv's (in cartesian coords)

$=: \Delta f$ "Laplace f"

shows up all over physics

in Eqns governing wave motion, heat conduction

electrostatics, ...