preliminary geometric meaning of div and curl:

curl in 2d (via the trick $\begin{bmatrix} F_1(x,y,\cancel{z}) \\ F_2(x,y,\cancel{z}) \\ 0 \end{bmatrix}$

If $\operatorname{curl} \vec{F} = 0$ then $\oint_C \vec{F}\cdot d\vec{s} = 0$

Imagine C a small curve about a point P

$$\oint_C \vec{F}\cdot d\vec{s} = \int_C (\text{tangential comp of } \vec{F})\, ds$$

Assume you put a little paddle-wheel at P

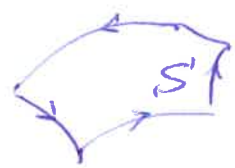
turning effect (not translation) caused by moments like $\vec{F}\cdot d\vec{s}$

The curl at P will give a net rotation rate of a paddle wheel at P under the influence of the velocity field $\vec{F}$

→ more detailed explanation in the next chapter, where we will prove

$$\underbrace{\int_{\text{surface } S} (\operatorname{curl} \vec{F})\cdot \vec{N}\, d\text{area}}_{\text{we have yet to discuss this}} = \oint_{\text{boundary of } S} \vec{F}\cdot d\vec{s}$$

normal unit vector



we have yet to discuss this

Also for div:

div $\vec{F} > 0$ at P if in net effect fluid is leaving P

(imagine $\vec{F}$ fluid velocity) (if more fluid is flowing away than coming towards P)

< 0 ... arriving at P

div $\vec{F} = 0$ is characteristic for velocity fields of incompressible fluids

Again the reason behind this interpretation lies in an integral theorem

Sneak preview: $$\int_W \operatorname{div}\vec{F}\, d\mathrm{vol} = \underbrace{\int_{\text{boundary of } W} \vec{F}\cdot\vec{N}\, d\mathrm{area}}_{\text{we have yet to discuss this}}$$

normal vector

That's ample motivation to study surfaces and integrals over them. (→ Sec 16.4)

Def of a parametrized surface:

Take $(u,v) \in D \subset \mathbb{R}^2$ for a domain D and consider a map

$\vec{r} : D \to \mathbb{R}^3$ ($\vec{r}$ should at least be differentiable)

the image of $\vec{r}$ will be a surface

Example 1:

$$\vec{r}: (\phi, \theta) \mapsto \begin{bmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{bmatrix}$$

Surface is (part of) the unit sphere

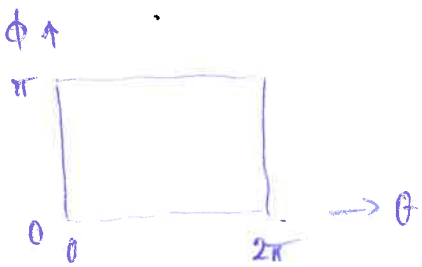
↓

"part of" if you take the open rectangle

$(\phi, \theta) \in$ rectangle

ϕ: $\frac{\pi}{2}$ - latitude

θ: longitude



Example 2:

$$\vec{r}: (x,y) \longmapsto \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$$

surface is the graph of f

Example 3: fix two numbers R, ρ $(R > \rho)$

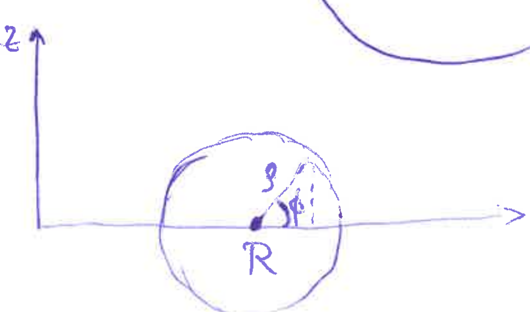
$$\vec{r}: (\phi, \theta) \longmapsto \vec{r}(\phi, \theta) = \begin{bmatrix} (R + \rho\cos\phi)\cos\theta \\ (R + \rho\cos\phi)\sin\theta \\ \rho\sin\phi \end{bmatrix}$$

$0 < \phi < 2\pi$

$0 < \theta < 2\pi$

foot print in (x,y) plane:

Cross section w/ (x,z)-plane: $\theta = 0$ s.t. $y = 0$



$x = R + \rho\cos\phi$

$z = \rho\sin\phi$

surface of a donut

If you have a param. surface

$$(u,v) \longmapsto \vec{r}(u,v)$$

and you fix one parameter (eg. $v = v_0$)

then $u \longmapsto \vec{r}(u, v_0)$ is a param. curve

$\frac{\partial}{\partial u}\vec{r}(u, v_0)$ is a tangent vector to this curve.

(and thus in particular tangent to the surface as well)

$\frac{\partial}{\partial v}\vec{r}(u,v)$ is another tangent vector

To ensure that neither of these vectors is $\vec{0}$ and that they are not parallel either, we

want $\left(\frac{\partial}{\partial u}\vec{r}\right) \times \left(\frac{\partial}{\partial v}\vec{r}\right) \neq \vec{0}$

This condition is what makes (u,v) "good" coordinates on the surface.