If you have a param. surface

$$(u,v) \longmapsto \vec{r}(u,v)$$

and you fix one parameter (eg. $v = v_0$)

then $u \mapsto \vec{r}(u, v_0)$ is a param. curve

$\frac{\partial}{\partial u} \vec{r}(u, v_0)$ is a tangent vector to this curve.

(and thus in particular tangent to the surface as well)

$\frac{\partial}{\partial v} \vec{r}(u,v)$ is another tangent vector

To ensure that neither of these vectors is $\vec{0}$ and that they are not parallel either, we

want $\left(\frac{\partial}{\partial u} \vec{r}\right) \times \left(\frac{\partial}{\partial v} \vec{r}\right) \neq \vec{0}$

This condition is what makes (u,v) "good" coordinates on the surface.

Now we do allow exceptions to this requirement on the boundary of the coordinate domain $D = \{\text{allowable } (u,v)\}$ in order to accommodate eg the poles of a sphere in spherical coord's

(We can get along with this quick-and-dirty exception b/c the exceptional points will be so few that omitting them would not affect the integral)

$\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ (being tangent vectors to the surface)

span the tangent plane.

Their cross product $\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ is a normal vector to the surface.

Example: sphere $\vec{r} = \begin{bmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ \cos\phi \end{bmatrix}$

$$\vec{r}_\phi = \begin{bmatrix} \cos\theta \cos\phi \\ \sin\theta \cos\phi \\ -\sin\phi \end{bmatrix} \qquad \vec{r}_\theta = \begin{bmatrix} -\sin\theta \sin\phi \\ \cos\theta \sin\phi \\ 0 \end{bmatrix}$$

$$\vec{N} = \vec{r}_\phi \times \vec{r}_\theta = \begin{bmatrix} \cos\theta \sin^2\phi \\ \sin\theta \sin^2\phi \\ \cos\phi \sin\phi \end{bmatrix}$$ which happens to be $= \sin\phi \cdot \vec{r}$

(as expected geometrically it goes in radial direction)

$\vec{N} \neq \vec{0}$ except when $\phi = 0$ or $\phi = \pi$ (North or South pole)

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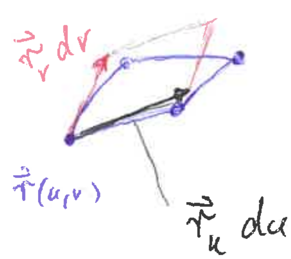
surface area, and surface element



$\vec{r}_v\, dv \approx \vec{r}(u, v+dv) - \vec{r}(u,v)$

The parallelogram spanned by $\vec{r}_u\, du$ and $\vec{r}_v\, dv$ is like a little fish scale approximating a corresponding piece of the surface area of such a scale: $\|\vec{r}_u\, du \times \vec{r}_v\, dv\| = \|\vec{r}_u \times \vec{r}_v\| du\, dv$

area of surface S is $\iint_D \underbrace{\|\vec{r}_u \times \vec{r}_v\|}_{\vec{N}} du\, dv$

$\|\vec{r}_u \times \vec{r}_u\| du\, dv =: dS$ surface area element

Now we can also define scalar surface integrals

$$\iint_S f(x,y,z)\, dS = \iint_D f(x(u,v), y(u,v), z(u,v))\, \|\vec{r}_u \times \vec{r}_v\|\, du\, dv$$

The usual applications are available:

- If f is a density (mass per unit area), we get the total mass
- In particular if $f \equiv 1$, we get the area
- If $f = (x^2+y^2)\overset{\text{density}}{\nu}$, we get a moment of inertia of the surface when rotating about z-axis
- $x_{CM} = \iint_S x\, dS \,/ \iint_S 1\, dS$

  $y_{CM} = \ldots\ y\ \ldots$ (ditto the rest)

  $z_{CM} = \ldots\ z\ \ldots$

An interesting special case

$$\vec{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ 0 \end{bmatrix}$$

(so this "fancy" surface is a piece of the plane!)

$$\vec{r}_u \times \vec{r}_v = \begin{bmatrix} 0 \\ 0 \\ x_u y_v - x_v y_u \end{bmatrix}$$

$$\|\vec{r}_u \times \vec{r}_v\| \, du\,dv = \underbrace{|x_u y_v - x_v y_u|}_{\text{Jacobi det.}} \, du\,dv$$

Surface integrals of vector fields (Sec 16.5)

(Flux integrals)