An interesting special case

$$\vec{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ 0 \end{bmatrix}$$

(so this "fancy" surface is a piece of the plane!)

$$\vec{r}_u \times \vec{r}_v = \begin{bmatrix} 0 \\ 0 \\ x_u y_v - x_v y_u \end{bmatrix}$$

$$\|\vec{r}_u \times \vec{r}_v\| \, du\,dv = \underbrace{|x_u y_v - x_v y_u|}_{\text{Jacobi det.}} \, du\,dv$$

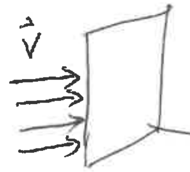
Surface integrals of vector fields (Sec 16.5)

(Flux integrals)

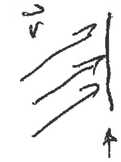
Interpretation: Suppose we interpret a vector field $\vec{v}$ as a velocity field of a fluid and want to know what amount of fluid per time passes through a surface S

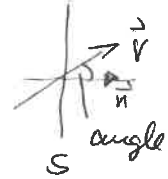
Approximate Riemann sum: Take small pieces of the surface S ("fish scales" considered last section) & pretend $\vec{v}$ is constant near those scales

In the limit of small fishscales get an integral.



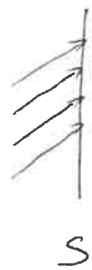
in a cross section





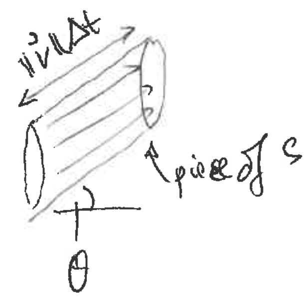
angle θ b/w $\vec{v}$ and the normal direction to S'

We'll fix a "side" of the surface by choosing a unit normal vector $\vec{n}$ (out of two choices)



within time Δt, each fluid particle moves by a vector ~~$\vec{v}\Delta t$~~ $\vec{v} \cdot \Delta t$

Hence, the volume of fluid passing through S is contained in an oblique cylinder, whose volume is



$$\underbrace{\Delta S}_{\text{base area}} \quad \underbrace{\|\vec{v}\| \Delta t \cdot \cos\theta}_{\text{height of cylinder}}$$

$$\|\vec{v}\| \cos\theta = \vec{v} \circ \vec{n}$$

pass through piece of S' within time Δt : $\Delta S \ \vec{v} \circ \vec{n} \ \Delta t$

$$\Delta S' \approx \|\vec{r}_u \times \vec{r}_v\| \, \Delta u \, \Delta v$$

$\vec{N} = \vec{r}_u \times \vec{r}_v$ is a normal vector ; $\vec{n} = \dfrac{\vec{N}}{\|\vec{N}\|} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$

Thus pass through piece of S within time Δt is

$$\vec{v} \cdot \frac{\vec{N}}{\|\vec{r}_u \times \vec{r}_v\|} \, \|\vec{r}_u \times \vec{r}_v\| \, \Delta u \Delta v \cdot \Delta t$$

What passes through all of S within time Δt is an integral

$$\left(\iint_S \vec{v} \cdot \vec{N} \, du \, dv \right) \Delta t = \left(\iint_S \vec{v} \cdot \vec{n} \, dS \right) \Delta t$$

flux through S ($:=$ volume of fluid / time) is

$$\iint_S \vec{v} \cdot \underbrace{\vec{n} \, dS}_{d\vec{S}} = \iint_S \vec{v} \cdot \underbrace{(\vec{r}_u \times \vec{r}_v) \, du \, dv}_{d\vec{S}}$$

(vector valued surface element)

This integral is called a ~~vector~~ surface integral of the vector field $\vec{v}$, aka flux integral.

Application

In electromagnetism: flux of the electric field $\vec{E}$ through a closed surface S = (proportionality constant) × total el. charge contained in the interior of S

Compare also sneak preview on page 106

$$\iiint_W \operatorname{div} \vec{F} \, dvol = \iint_{\text{bdry} W} \vec{F} \cdot d\vec{S}$$

(Will give statement that $\operatorname{div}(\text{electric field } \vec{E}) = (\text{const}) \cdot \text{charge density}$)

Note: for a flux integral through S to make sense, it is necessary that S be <u>orientable</u>, i.e., that it possible to choose a unit normal vector $\vec{n}$ on S continuously all over S.

Eg: • If S is a sphere  we can decide to choose $\vec{n}$ always "outwards pointing"

Alternatively we could choose $\vec{n}$ always inward-pointing

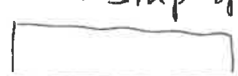
These two choices are called "orientations of S")

• Any S that is the boundary of some solid (3 dim) object is orientable

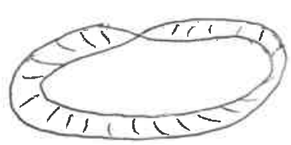
By convention we choose $\vec{n}$ outward-pointing for such surfaces

• If S is a graph $z = f(x,y)$, we can either choose $\vec{n}$ upwards pointing $(n_3 > 0)$ or downwards pointing $(n_3 < 0)$

• Möbius band: take a strip of paper



twist one end by 180° and glue ends together



this surface is <u>not</u> orientable.

• If S <u>is</u> orientable and we change the orientation, then the flux integral picks up a minus sign

Calculational example:

Flux of $\begin{bmatrix} x \\ y+z \\ 2z-x \end{bmatrix}$ through sphere of radius 2 (outwards orientation)

$$S: \quad \vec{r} = \begin{bmatrix} 2\sin\phi\cos\theta \\ 2\sin\phi\sin\theta \\ 2\cos\phi \end{bmatrix}$$

$$\vec{r}_\phi \times \vec{r}_\theta = 4\sin\phi \begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix} \qquad \text{(points outward)}$$

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$$\int_0^{2\pi}\int_0^{\pi} \underbrace{\begin{bmatrix} 2\sin\phi\cos\theta \\ 2\sin\phi\sin\theta + 2\cos\phi \\ 4\cos\phi - 2\sin\phi\cos\theta \end{bmatrix}}_{\vec{F}(\vec{r}(\phi,\theta))} \cdot \underbrace{4\sin\phi \begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix} d\phi\, d\theta}_{d\vec{S}}$$

= [... routine algebra ...]