Calculational example:

Flux of $\begin{bmatrix} x \\ y+z \\ 2z-x \end{bmatrix}$ through sphere of radius 2 (outwards orientation)

$$S: \quad \vec{r}_{\phi} = \begin{bmatrix} 2\sin\phi \, \cos\theta \\ 2\sin\phi \, \sin\theta \\ 2\cos\phi \end{bmatrix}$$

$$\vec{r}_\phi \times \vec{r}_\theta = 4\sin\phi \begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix} \qquad \text{(points outward)}$$

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$$\int_0^{2\pi}\int_0^{\pi} \underbrace{\begin{bmatrix} 2\sin\phi\cos\theta \\ 2\sin\phi\sin\theta + 2\cos\phi \\ 4\cos\phi - 2\sin\phi\cos\theta \end{bmatrix}}_{\vec{F}(\vec{r}(\phi,\theta))} \cdot \underbrace{4\sin\phi \begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix} d\phi \, d\theta}_{d\vec{S}}$$

$$= [\ldots \text{ routine algebra} \ldots] = \frac{128\pi}{3}$$

Let me return to the sneak previewed formula. It claimed

$$\underbrace{\iiint_{\text{Ball of radius 2}} \operatorname{div}\vec{F} \, dV}_{} = \iint_{\text{Sphere of radius 2}} \vec{F}\cdot d\vec{S}$$

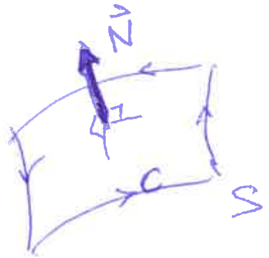
$\operatorname{div}\vec{F} = 1 + 1 + 2 = 4$

Thus $= 4 \cdot$ vol of ball of radius 2 $= 4 \cdot 2^3 \cdot \frac{4\pi}{3} = \frac{128\pi}{3}$

confirming the promised formula.

Application in Electrodynamics:

If C is the boundary of surface S, with appropriate orientations:

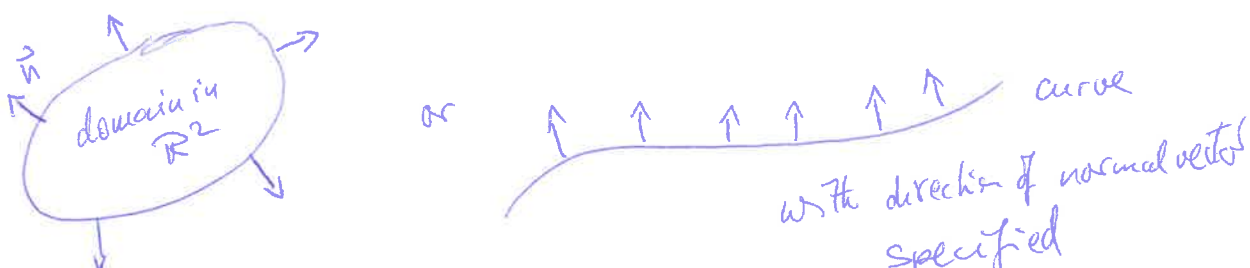


matching orientations follow the right hand hitchhiker thumb rule

~~the~~ and $\vec{B}$ is the magnetic field (changing with time)
and $\vec{E}$ is the electric field

Then $\displaystyle\oint_C \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{S}$ (proportionality constant may occur depending on choice of units)

We can also do flux integrals in a planar setting



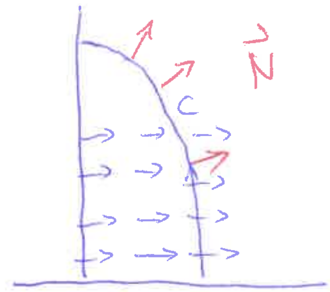
These flux integrals will be vector line integrals (Example 9 in Sec 16.2)

$\displaystyle\int \vec{V}\cdot \underbrace{\vec{n}\,ds}_{\vec{N}\cdot dt}$ $\qquad$ $\vec{n}$ unit normal vector $\quad \underbrace{ds}_{\|\vec{r}\,'(t)\|\,dt}$ length element

$\vec{N} =$ " $\vec{r}\,'(t)$ rotated by $90°$ "

curve C:

$$\vec{v} = \begin{bmatrix} 3+2y-y^3/3 \\ 0 \end{bmatrix} \qquad \vec{r}(t) = \begin{bmatrix} 3\cos t \\ 6\sin t \end{bmatrix} \qquad 0 \le t \le \frac{\pi}{2}$$



(lengths of vec's not to scale)

$$\vec{r}\,'(t) = \begin{bmatrix} -3\sin t \\ 6\cos t \end{bmatrix} \qquad \vec{N}(t) = \begin{bmatrix} 6\cos t \\ 3\sin t \end{bmatrix}$$

$$\int \vec{v}\cdot\vec{N}\,dt = \int_0^{\pi/2} \begin{bmatrix} 3 + 12\sin t - \frac{216\sin^3 t}{3} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 6\cos t \\ 3\sin t \end{bmatrix} dt$$

$$= \left[\text{easy algebra}\right] = \dots 30$$

If D is a domain in the plane, then its boundary is a curve (or maybe several curves). We denote this boundary as ∂D (the same curly ∂ that we used for partial deriv's is also used for boundary!)

Same applies if W is a solid body (with its bdry a surface, or possibly several), then this bdry is also called ∂W

Consistent orientation is understood when we use the ∂ notation for bdry