Green's Thm (Ch 17.1)

If D is a (reasonable) planar domain with ∂D its boundary and $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ $\underbrace{\text{a vector field}}_{\text{cont. differentiable}}$ on $\bar{D} = D \cup \partial D$, then

$$\oint_{\partial D} \underbrace{F_1\,dx + F_2\,dy}_{\vec{F} \cdot d\vec{s}} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

Where ∂D is oriented such that D is "on the left".

Example: $\vec{F} = \begin{bmatrix} x^2 y \\ x - y^2 \end{bmatrix}$ D = unit disc $\vec{r}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$

$$\int_{\partial D} F_1\,dx + F_2\,dy = \int_0^{2\pi} \left(\cos^2 t \sin t\,(-\sin t) + (\cos t - \sin^2 t) \cdot \cos t\right) dt$$

$$= \underbrace{-\int_0^{2\pi} \underbrace{\cos^2 t \sin^2 t}_{(\frac{1}{2}\sin 2t)^2}\,dt}_{-\frac{\pi}{4}} + \underbrace{\int_0^{2\pi} \cos^2 t\,dt}_{\pi} - \underbrace{\int_0^{2\pi} \cos t \sin^2 t\,dt}_{0}$$

$$= \frac{3\pi}{4}$$

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_D (1 - x^2)\,dA \underset{\text{in polar coords}}{=} \int_0^{2\pi}\int_0^1 (1 - r^2\cos^2\varphi)\,r\,dr\,d\varphi$$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{4}\cos^2\varphi\right) d\varphi = \pi - \frac{1}{4}\pi = \frac{3\pi}{4}$$

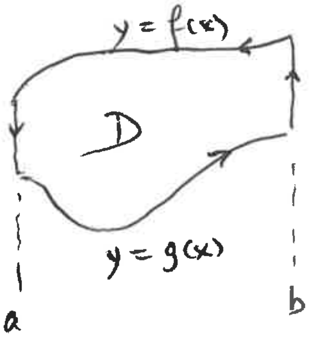
We prove Green's thm by separate calculations:

$$(1) \quad \oint_{\partial D} F_1\,dx = -\iint_D \frac{\partial F_1}{\partial y}\,dA$$

$$(2) \quad \oint_{\partial D} F_2\,dy = \iint_D \frac{\partial F_2}{\partial x}\,dA$$

I'll just do (1). The calc for (2) is analogous.
It will use the fund'l thm of calculus - $\int \Big(\underbrace{\int \frac{\partial F_1}{\partial y}\,dy}_{F_1(\text{upper end}) - F_1(\text{lower end})} \Big)\,dx$

First assume D is vertically simple



$$\oint_{\partial D} F_1\,dx = \int_{x=a}^{x=b} F_1(x, g(x))\,dx + 0$$

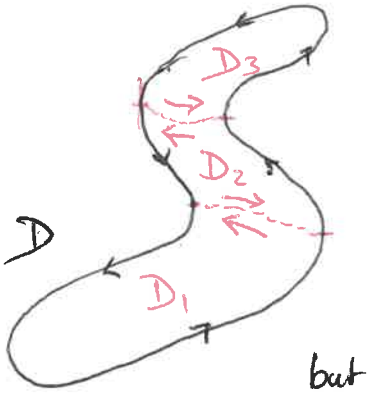
$$+ (-1)\cdot \int_a^b F_1(x, f(x))\,dx + 0$$

$$= -\int_a^b \Big(F_1(x, f(x)) - F_1(x, g(x))\Big)\,dx$$

$$= -\int_a^b \left(\int_{g(x)}^{f(x)} \frac{\partial F_1}{\partial y}(x, y)\,dy \right) dx$$

$$= -\iint \frac{\partial F_1}{\partial y}\,dA$$

What if D is not vertically simple (but can be decomposed into vertically simple domains)?



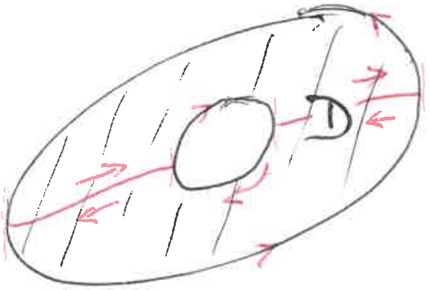
easy: $\iint\limits_{D} \cdots dA = \iint\limits_{D_1} \cdots dA + \iint\limits_{D_2} \cdots dA + \iint\limits_{D_3} \cdots dA$

but surprise: $\cancel{\oint} \oint\limits_{\partial D} \vec{F}\cdot d\vec{s} = \oint\limits_{\partial D_1} \vec{F}\cdot d\vec{s} + \oint\limits_{\partial D_2} \vec{F}\cdot d\vec{s} + \int\limits_{\partial D_3} \vec{F}\cdot d\vec{s}$

(understood with $\vec{F} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix}$)

black parts of $\partial D_1, \partial D_2, \partial D_3$ assemble nicely to ∂D
but what about the red parts? – They cancel out pairwise
(each red part shows up twice, with opposite orientation

Works also if D has "holes":



This takes care of (1). To do (2), use a similar calc
(D horizontally simple, or decomposable into hor. simple parts)

Some parts of Green's thm, we already know:

Eg if $\vec{F} = \vec{\nabla} f$ then $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$ (by Clairaut)

ie $\iint\limits_{D} \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right) dA = 0$

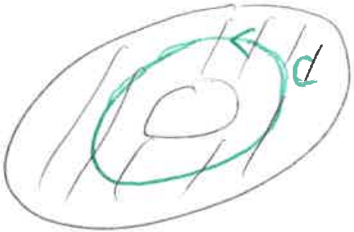
Then Green's thm tells us (what we already know) $\oint\limits_{\partial D} \underset{\substack{\| \\ \nabla f}}{\vec{F}}\cdot d\vec{s} = 0$

Conversely if $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$, can we use that alone,

to show that $\oint_C \vec{F} \cdot d\vec{s} = 0$ for any C ?

Well, only if C is the bdry ∂D of some D

In a non-simply connected domain, that may not be the case



But in simply connected domains, we can find such a domain whose boundary is C and then we prove

that indeed [illegible] "cross deriv coincide and domain simply connected $\Longrightarrow$ vf conservative"