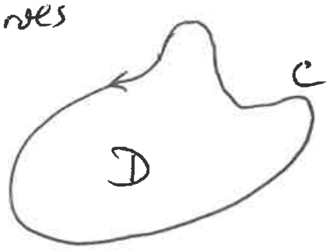
An application of Green's thm:

$$\oint_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

We can use this formula to calculate area enclosed by parametrized curves



param. of C given, but want to know area inside

Example: $x(t) = -1 + 2\cos t - \cos 2t$ cardiod $0 \le t \le 2\pi$

$y(t) = 2\sin t - \sin 2t$

$C = \partial D$

to integrate $\iint_D 1 \, dA$ (which is area (D)) via Green's Thm,

we need to invent some $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ such that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$

Many options: $\vec{F} = \begin{bmatrix} 0 \\ x \end{bmatrix}$, or $\vec{F} = \begin{bmatrix} -y \\ 0 \end{bmatrix}$,

or $\vec{F} = \frac{1}{2} \begin{bmatrix} -y \\ x \end{bmatrix}$

For cardiod, we take

$$\text{area (cardioid interior)} = -\oint_{\partial \text{ cardiod}} y\, dx = -\int_0^{2\pi} (2\sin t - \sin 2t)(-2\sin t + 2\sin 2t)\, dt$$

$$= \int_0^{2\pi} (4\sin^2 t + 2\sin^2 2t - 6\sin t \sin 2t)\, dt$$

$$= 4\pi + 2\pi - 0 = \underline{\underline{6\pi}}$$

Didn't really need(*) to understand the detailed shape of C to do this calc. Without Green, we would have needed such insight in order to split domain into (eg) vertically simple pieces.

(*) but there is a caveat; we do use that C indeed bounds a domain D that lies to the left of it ($C = \partial D$); in particular that C doesn't self-intersect

C

called a lemniscate

If you calc $-\oint_C y\,dx$,

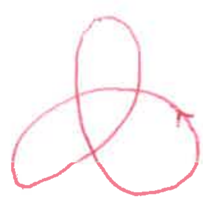
you will not get area , but rather

area  + (-1) · area 

so you'll get 0

Challenge: Think what area you'd get if you ~~do it~~ calc $-\oint_C -y\,dx$ for

C



In Hwk you invent $\vec{F}$ to calc $\int_D x\,dA$, $\int_D y\,dA$ via Green

Previously, I had promised you two "big" theorems

(1) $$\oint_{\partial S} \vec{G}\cdot d\vec{s} = \iint_S \operatorname{curl}\vec{G}\cdot d\vec{S}$$ (Ch 17.2) S surface in space

(2) $$\iint_{\partial W} \vec{G}\cdot d\vec{S} = \iiint_W (\operatorname{div}\vec{G})\, dVol$$ (Ch 17.3) W body in space

(2) has a variant for domains in the plane

(2a) $$\int_{\partial D} \vec{G}\cdot\vec{n}\, ds = \iint_D (\operatorname{div}\vec{G})\, darea$$ D domain in the plane

Green's theorem in a sense spawns all of these.

The special case of (1) when S happens to lie in the (x,y) plane is nothing but Green's thm, when $\vec{G} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix}$ $\quad d\vec{S} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} darea$

$$\oint_{\partial S} \vec{G}\cdot d\vec{s} = \oint_{\partial S} F_1\,dx + F_2\,dy \cancel{+0\cdot dz}$$

$$\operatorname{curl}\vec{G} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{bmatrix}$$

Version (2a) is also immediate from Green, if we take

$$\vec{G} = \begin{bmatrix} F_2 \\ -F_1 \end{bmatrix}$$ Then $\operatorname{div}\vec{G} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

$$\iint_D \operatorname{div}\vec{G}\, dA = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$ the rhs of Green's thm

Let's look at $\oint_{\partial D} \vec{G} \cdot \vec{n}\, ds$

If ∂D is parametrised as $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

then the tangent vector is $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$

and the tangential length element $d\vec{s} = \begin{bmatrix} x' \\ y' \end{bmatrix} dt$

rotating $\begin{bmatrix} x' \\ y' \end{bmatrix}$ by 90° ~~counter~~ clockwise

obtains $\vec{N} = \begin{bmatrix} y' \\ -x' \end{bmatrix}$

$$\vec{n}\, ds = \begin{bmatrix} y' \\ -x' \end{bmatrix} dt = \begin{bmatrix} dy \\ -dx \end{bmatrix}$$

$$\oint_{\partial D} \vec{G} \cdot \vec{n}\, ds = \oint_{\partial D} F_2\, dy - F_1 \cdot (-dx) = \oint_{\partial D} F_1 dx + F_2 dy$$

indeed the left side of Green's theorem

Now let's look at (1) for general surfaces in space

Refer to Chap 17.2 for some pix of surfaces S and how, using the normal vector $\vec{n}$ on the oriented surface S, we obtain the proper orientation of each component curve (if any) of ∂S; namely walking along ∂S (heads "up" in $\vec{n}$ direction), S should be to the left of the walker...