Goal is to prove Stokes' Thm

$$\cancel{\iint_S \vec{F}\cdot d\vec{S} = \oint_{\partial S} (\mathrm{curl}\,\vec{F})\cdot d\vec{s}} \qquad \iint_S (\mathrm{curl}\,\vec{F})\cdot d\vec{S} = \oint_{\partial S} \vec{F}\cdot d\vec{s}$$

<u>BTW:</u> these flux (and scalar surface) integrals do not depend on the parametrization on S

We will assume for simplicity that S is a graph of a fct over a planar domain. (eg $z=f(x,y)$, or $y=g(x,z)$, or $x=h(y,z)$)

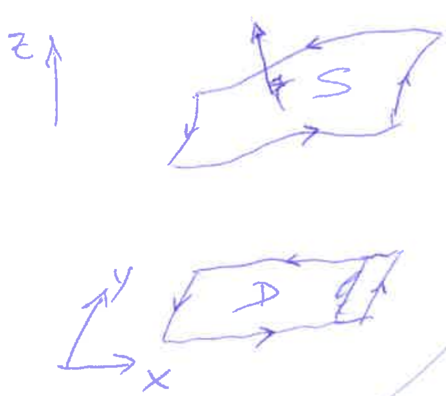
The general case would follow by taking S apart into pieces $S_1, S_2, \ldots$ that are graphs

$$\oint_{\partial S} \ldots = \oint_{\partial S_1} \ldots + \oint_{\partial S_2} \ldots + \ldots$$

b/c pieces where S_1, S_2 meet will have contrib's to $\oint_{\partial S_1}$, $\oint_{\partial S_2}$ that cancel out

So let $S: \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$ (other graphs like $y=g(x,z)$ are done exactly alike)

$(x,y)\in D$



We also decompose the vector field

$$\vec{F} = \begin{bmatrix} F_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ F_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F_3 \end{bmatrix}$$

and will study $\begin{bmatrix}F_1\\0\\0\end{bmatrix}$. F_2 is almost identical calc. F_3 is quite similar → HW

Stokes for the whole $\vec{F}$ is obtained as a sum of the three

So we want to show

$$(*)\qquad \iint_{\text{graph} f} \underbrace{\text{curl}\begin{bmatrix}F_1\\0\\0\end{bmatrix}\cdot \cancel{ds}\, d\vec{S}} = \oint_{\partial(\text{graph} f)} \left(\cancel{\text{curl}}\begin{bmatrix}F_1\\0\\0\end{bmatrix}\right)\cdot d\vec{s}$$

need to get $d\vec{S}$ where $S: \begin{bmatrix}x\\y\\f(x,y)\end{bmatrix}$

$$\vec{T}_x \times \vec{T}_y \; dx\,dy$$

$$= \begin{bmatrix}1\\0\\\partial f/\partial x\end{bmatrix} \times \begin{bmatrix}0\\1\\\partial f/\partial y\end{bmatrix} dx\,dy = \begin{bmatrix}-\partial f/\partial x\\-\partial f/\partial y\\1\end{bmatrix}. \quad \text{Also } \text{curl}\begin{bmatrix}F_1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\+\partial F_1/\partial z\\-\partial F_1/\partial y\end{bmatrix}$$

The left hand side of (*) is therefore

$$-\iint_D \left(F_{1,z}(x,y,f(x,y))\,\frac{\partial f}{\partial y}(x,y)\,\cancel{dx\,dy} + F_{1,y}(\dots)\right) dx\,dy \qquad \text{(lhs of Stokes eval'd)}$$

The right side of (*) is

$$\oint_{\partial D} \begin{bmatrix}F_1\\\cancel{\partial F_1/\partial z}\;0\\\cancel{-\partial F_1/\partial y}\;0\end{bmatrix}(x,y,f(x,y)) \cdot \begin{bmatrix}\cancel{dx}\\\cancel{dy}\\\cancel{dz}\end{bmatrix}\begin{bmatrix}x'\\y'\\f_x\cdot x' + f_y\cdot y'\end{bmatrix} dt \qquad \left\{\begin{array}{l}x = x(t)\\y = y(t)\\z = f(x(t),y(t))\end{array}\right. \quad a \le t \le b.$$

∂D

//

(rhs of Stokes eval'd)

$$= \int_a^b x'(t)\, \cancel{y'(t)}\; F_{1}(x(t),y(t),f(x(t),y(t)))\,dt \;\cancel{+ F_{1,y}(\dots)\left(f_x(x(t),y(t))x'(t) + f_y(\dots)\,y'(t)\right)} dt$$

$$= \oint \cancel{F_1\,dx}\; F_1(x,y,f(x,y))\,dx \;\cancel{\left(F_{1,z} - F_{1,y} f_y\right)dy - F_{1,y} f_x\,dx}$$

Green's thm

for $G_1(x,y) = F_1(x,y,f(x,y))$ $\qquad G_2 = 0$

$$\cancel{\oint_{\partial D}\left(\frac{\partial G_1}{\partial y} + \frac{\partial G_2}{\partial x}\right) = \iint_D}$$

$$\iint_D \left(-\frac{\partial G_1}{\partial y} + \frac{\partial G_2}{\partial x}\right) dx\,dy = \oint_{\partial D} G_1\,dx + G_2\,dy$$

$$\frac{\partial G_1}{\partial y} = F_{1,y}\cdot 1 + F_{1,z}\cdot\frac{\partial f}{\partial y} \qquad \text{by chain rule}$$

Thus $\iint_D -\frac{\partial G_1}{\partial y}\,dx\,dy$ is exactly the lhs of (*) from last page

Likewise $\oint_{\partial D} G_1\,dx$ is exactly the rhs of (*) from last page

We say $\vec{F}$ has a vector potential $\vec{A}$ if $\vec{F} = \text{curl}\,\vec{A}$

(analogy: $\vec{F}$ has a (scalar) potential f if $\vec{F} = \vec{\nabla} f$)

~~Not~~ Not every $\vec{F}$ has a vector potential

For $\vec{F}$ to have a vector potential, it is necessary that $\text{div}\,\vec{F} = 0$
b/c $\text{div curl}\,\vec{A} = 0$)

(analogy: For $\vec{F}$ to have a scalar potential f, it is necessary that
$\text{curl}\,\vec{F} = \vec{0}$ b/c $\text{curl}\,\nabla f = \vec{0}$)

If $\vec{F}$ has a vector potential, then the flux $\iint_S \vec{F} \cdot d\vec{S}$
depends only on the boundary of S, not S itself

b/c $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \text{curl}\,\vec{A} \cdot d\vec{S} = \oint_{\partial S} \vec{A} \cdot d\vec{s}$

(analogy: If $\vec{F}$ has a scalar potential then the line integral
$\int_C \vec{F} \cdot d\vec{s}$ depends only on the endpoints of C, not the curve C itself)

Vector potentials are vastly non-unique.

If $\vec{A}$ is a vect. potential for $\vec{F}$ then $\vec{A} + \nabla f$ (any fct f)
also is a vector potential b/c $\text{curl}\,\vec{A} + \nabla f = \text{curl}\,\vec{A} + \underbrace{\text{curl}\,\nabla f}_{0}$

In electrodyn: $\vec{B}$ magnetic field (which does satisfy $\text{div}\,\vec{B} = 0$)
has a vector potential $\vec{B} = \text{curl}\,\vec{A}$

$\rightarrow$ may read up on physics of this in Ch 17.2