Vector-Valued functions

(Sec 13.1)

We are here still talking about single-variable functions, which assign a value $f(t)$ to a <u>real number</u> t. However, we allow the values $f(t)$ to be (points or) vectors. Typical situation: a particle moving in space: the position of the particle at time t is given as a point in $\mathbb{R}^3$ by coordinates $(x(t), y(t), z(t))$.

We can also represent this point by its position vector, i.e. the vector from the origin to the point. That position vector is, in coordinates,

$$\begin{bmatrix} x(t)-0 \\ y(t)-0 \\ z(t)-0 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}.$$

We often call such a position vector as $\vec{r}(t)$, rather than $\vec{f}(t)$.

<u>Examples:</u> particle moves on a certain circle above xy-plane

$$r(t) = \begin{bmatrix} \cos t \\ \sin t \\ 1 \end{bmatrix}, \text{ or:}$$

$$r(t) = \begin{bmatrix} \cos 3t \\ \sin 3t \\ 1 \end{bmatrix}$$ (<u>same</u> circle but moving 3 times as fast)

Or a helix (spiral staircase)

$$r(t) = \begin{bmatrix} 2\cos t \\ 2\sin t \\ 3t \end{bmatrix}$$

Language: The path (or parametrized curve) refers to the entire function, assigning to each t the point whose position vector is $\vec{r}(t)$.

The curve is the collection of points traced out by $r(t)$ as t varies, but without regard to which point belongs to which t. Thus, above

$$\begin{bmatrix} \cos t \\ \sin t \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \cos 3t \\ \sin 3t \\ 1 \end{bmatrix}$$ represent different paths,

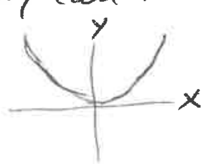
but the curves are the same

It is often not easy to see what curve is traced out by a particular vector-valued function, even in simple cases like

$$\vec{r}(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}.$$ Computer graphic systems that allow you to plot the curve and at the same time change the vantage point, help. Or you look at projections into the coordinate planes, eg:

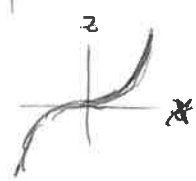
proj into x,y plane (architects may call this the "footprint")

$$\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$$ a parabola $y = x^2$



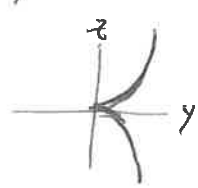
proj. into x,z plane ("front view"?)

$x = t$
$z = t^3$



proj into y,z plane ("side view"?)

$y = t^2$
$z = t^3$



Ch. 13.1 has examples of curves that arise as intersections of surfaces; I'll skip these examples for now.

Easy hwk pblms (due Thu <u>next</u> week) #4, 5, 8, 10, 34

Note the distinction in #34: When we ask whether the <u>paths</u> $r_1(t)$ and $r_2(t)$ <u>collide</u>, we ask whether particle 1 and particle 2 are at the same location for some time t.
When we ask whether these <u>curves</u> <u>intersect</u>, we ask whether there is a point that is visited both by particle 1 at some time t and by particle 2 at some (possibly different time) s.

(Sec 13.2)

Now let's do calculus with such paths: eg derivatives are defined as

$$\vec{r}\,'(t) := \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \qquad \text{(same as in Calc 1)}$$

The difference of vectors is calculated componentwise (and so are limits)

eg if $\vec{r}(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$, then

$$\frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \frac{\begin{bmatrix} t+h \\ (t+h)^2 \\ (t+h)^3 \end{bmatrix} - \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}}{h} = \begin{bmatrix} \frac{(t+h)-t}{h} \\ \frac{(t+h)^2-t^2}{h} \\ \frac{(t+h)^3-t^3}{h} \end{bmatrix}$$

$$\vec{r}\,'(t) = \lim_{h\to 0} \begin{bmatrix} \\ \\ \end{bmatrix} := \begin{bmatrix} \lim\limits_{h\to 0} \frac{t+h-t}{h} \\ \lim\limits_{h\to 0} \frac{(t+h)^2-t^2}{h} \\ \lim\limits_{h\to 0} \frac{(t+h)^3-t^3}{h} \end{bmatrix} = \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix}$$

Conclusion: We calculate derivatives of vector valued functions by taking the derivative of each component.

I will explain ~~later~~ soon why the following simple facts are true:

(1) If $\vec{r}(t)$ gives the position vector of a particle at time t, then the derivative $\vec{r}\,'(t)$ gives the velocity vector at time t.

(2) $\vec{r}\,'(t)$ is tangential to the curve traced out by $\vec{r}(t)$

(see Sec 13.3 for these)

But first let's harvest the usual rules for derivatives:

$$\frac{d}{dt}\left(\vec{r}_1(t)+\vec{r}_2(t)\right) = \vec{r}_1\,'(t) + \vec{r}_2\,'(t)$$ (Deriv's of sums are calc'd term by term)

$$\frac{d}{dt}\left(c\cdot\vec{r}(t)\right) = c\cdot\vec{r}\,'(t)$$ for c a constant.

This is because addition of vectors & mult. of vectors by a number are done componentwise. So we get these rules directly from the same rules for the scalar valued case in Calc 1.

Product rule:

$$\frac{d}{dt}\left(f(t)\cdot\vec{r}(t)\right) = f'(t)\cdot\vec{r}(t) + f(t)\cdot\vec{r}\,'(t)$$