More product rules:

$$\frac{d}{dt}\left(\vec{r}_1(t)\bullet\vec{r}_2(t)\right) = \vec{r}_1{}'(t)\bullet\vec{r}_2(t) + \vec{r}_1(t)\bullet\vec{r}_2{}'(t)$$

$$\frac{d}{dt}\left(\vec{r}_1(t)\times\vec{r}_2(t)\right) = \vec{r}_1{}'(t)\times\vec{r}_2(t) + \vec{r}_1(t)\times\vec{r}_2{}'(t)$$

Chain rule:

$$\frac{d}{dt}\vec{r}(f(t)) = \vec{r}\,'(f(t))\cdot f'(t)$$

How come?

eg chain rule

$$\frac{d}{dt}\vec{r}(f(t)) = \frac{d}{dt}\begin{bmatrix} x(f(t)) \\ y(f(t)) \\ z(f(t)) \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}x(f(t)) \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} x'(f(t))\cdot f'(t) \\ \vdots \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} x'(f(t)) \\ y'(f(t)) \\ z'(f(t)) \end{bmatrix}\cdot f'(t) = \vec{r}\,'(f(t))\cdot f'(t)$$

Dot product : similar

$\times$ product : -"-

Component-free proof of product rule for cross product:

$$\frac{d}{dt}\left(\vec{r}_1(t)\times\vec{r}_2(t)\right) = \lim_{h\to 0}\frac{\vec{r}_1(t+h)\times\vec{r}_2(t+h) - \vec{r}_1(t)\times\vec{r}_2(t)}{h} =$$

$$= \lim_{h\to 0}\frac{\vec{r}_1(t+h)\times\vec{r}_2(t+h) - \vec{r}_1(t)\times\vec{r}_2(t+h) + \vec{r}_1(t)\times\vec{r}_2(t+h) - \vec{r}_1(t)\times\vec{r}_2(t)}{h}$$

$$= \lim_{h\to 0} \left(\frac{\vec{r}_1(t+h) - \vec{r}_1(t)}{h} \times \vec{r}_2(t+h) + \vec{r}_1(t) \times \frac{\vec{r}_2(t+h) - \vec{r}_2(t)}{h} \right)$$

$$= \lim_{h\to 0} \frac{\vec{r}_1(t+h) - \vec{r}_1(t)}{h} \times \lim_{h\to 0} \vec{r}_2(t+h) + \cancel{\lim_{h\to 0}} \vec{r}_1(t) \times \lim_{h\to 0} \frac{\vec{r}_2(t+h) - \vec{r}_2(t)}{h}$$

$$= \vec{r}_1\,'(t) \times \vec{r}_2(t) + \vec{r}_1(t) \times \vec{r}_2\,'(t)$$

Examples:

Suppose $\vec{r}(t)$ is a vector val'd fct that satisfies

$$\|\vec{r}(t)\|^2 = 1$$

Then $\frac{d}{dt}\|\vec{r}(t)\|^2 = \frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t)$

$$= 2\,\vec{r}(t)\cdot\vec{r}\,'(t)$$

on the other hand, this is 0 b/c $\frac{d}{dt} 1 = 0$

We find, if $\|\vec{r}\|$ is constant, that $\vec{r}\,' \perp \vec{r}$

Example: $\frac{d}{dt}(\vec{r} \times m\vec{r}\,') = \vec{r}\,' \times m\vec{r}\,' + \vec{r} \times m\vec{r}\,''$

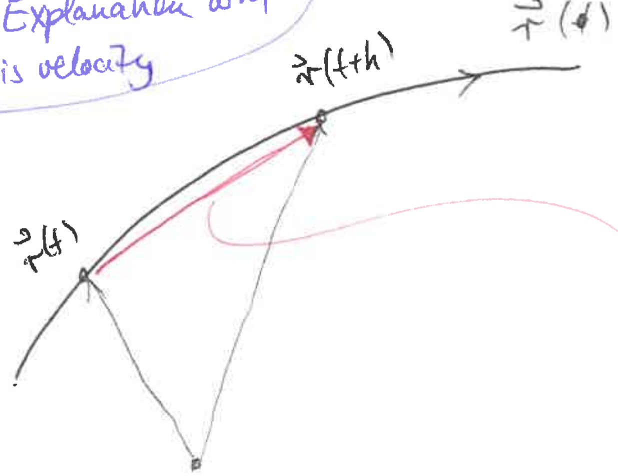
$$= \vec{r} \times m\vec{r}\,''$$

$\vec{r} \times$ (momentum $m\vec{r}\,'$) called angular momentum

$\vec{r} \times$ (force $m\vec{r}\,''$) called torque

$\frac{d}{dt}$ angular mom. $=$ torque

Explanation why $\vec{r}\,'$ is velocity



$$\frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

"average velocity" btw the two points

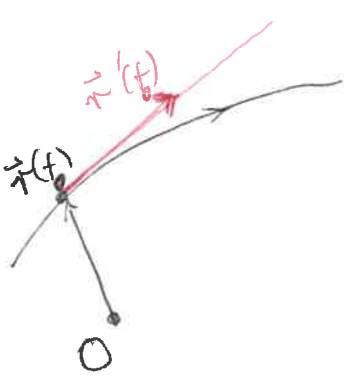
~~force~~ ("crow-fly" velocity)

$\vec{r}\,'(t) = \lim$ –"– = velocity of particle at time t (& thus at location $\vec{r}(t)$)

direction of $\vec{r}\,'(t)$: tangential to the curve

magnitude $\|\vec{r}\,'(t)\|$: speed of the particle

Sometimes it's nice to have an equation for the tangent line to a curve.



a point on the tangent line to the curve at $\vec{r}(t_0)$ will have position vector

$$\vec{r}(t_0) + t\,\vec{r}\,'(t_0)$$

may call it a different letter (eg s)