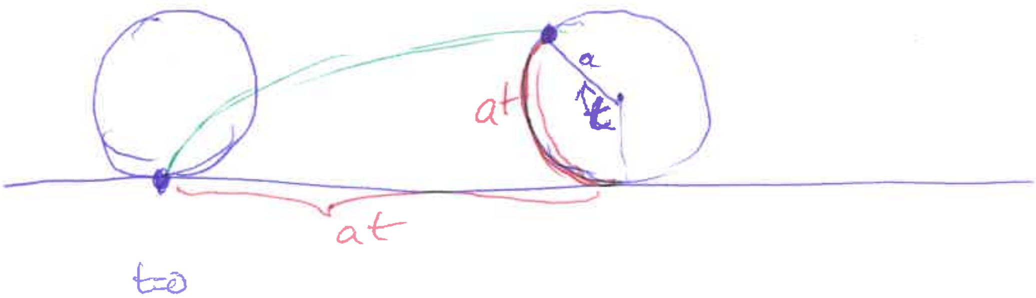
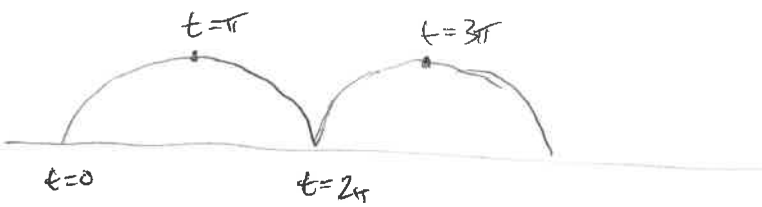
Example: Cycloid

circle of radius a, rolling on line



$$x = at - a\sin t$$
$$y = a - a\cos t$$

Will let $a=1$ $\qquad \vec{r}(t) = \begin{bmatrix} t-\sin t \\ 1-\cos t \end{bmatrix}$ cycloid



$$\vec{r}\,'(t) = \begin{bmatrix} 1-\cos t \\ \sin t \end{bmatrix}$$

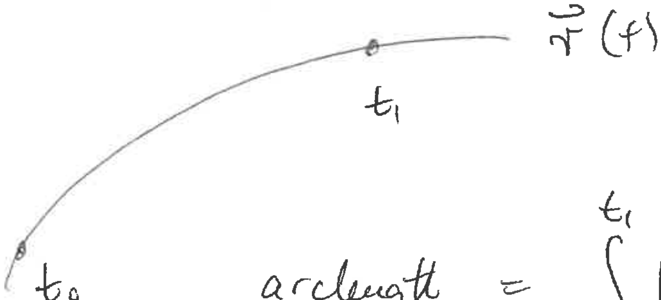
$$\vec{r}\,'(0) = \vec{r}\,'(2\pi) = \dots = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Observation (true in general)

If $\vec{r}(t)$ is differentiable and $\vec{r}\,'(t_0) \neq 0$ then the curve traced out by $\vec{r}(t)$ has a tangent vector $\vec{r}\,'(t_0)$ at point $\vec{r}(t_0)$ (and with it a tangent line). Whenever $\vec{r}\,'(t_0) = \vec{0}$, a tangent line to the curve may or may not exist.

Next question: How long is that arc of a cycloid, actually?

Arclength:



$$\underset{\text{btw } t_0 \,\&\, t_1}{\text{arclength}} = \int_{t_0}^{t_1} \|\vec{r}\,'(t)\|\,dt$$

(more precisely: distance traveled)

$\vec{r}\,'(t)$ velocity

$\|\vec{r}\,'(t)\|$ speed

$$\int_{t_0}^{t_1} \text{speed}(t)\,dt = \text{distance traveled.}$$

BTW if you want an explanation in terms of Riemann sums,
see Sec 11.2 of the book

Often, it's convenient to (re-)parametrize a curve by arclength.

If $\vec{r}(t)$ is a param. curve and you let $t = f(s)$

then the composite fct $\vec{r}(f(s))$ is another param. curve

tracing out the same geometric curve

If s is arclength (counted from some start pt), then we have
parametrized curve by arclength

Ex: Cycloid

$$\vec{r}(t) = \begin{bmatrix} t - \sin t \\ 1 - \cos t \end{bmatrix} \qquad \vec{r}\,'(t) = \begin{bmatrix} 1 - \cos t \\ \sin t \end{bmatrix}$$

$$\|\vec{r}\,'(t)\| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}$$

$$= \sqrt{4 \sin^2 \tfrac{t}{2}} = 2 \sin \tfrac{t}{2}$$

as ~~long~~ as $t \in [0, 2\pi]$ (else need $|\cdot|$)

$$s = \int_0^t 2 \sin \tfrac{t'}{2} \, dt' = 4\left(1 - \cos \tfrac{t}{2}\right) = 8 \sin^2 \tfrac{t}{4}$$

as $t = 2\pi$, after one roll, $s = 8 \sin^2 \frac{\pi}{2} = 8$

From $s = 8 \sin^2 \frac{t}{4}$, we can solve

$$t = 4 \arcsin \sqrt{\tfrac{s}{8}}$$

Reparam of cycloid by arclength

$$\vec{r}(t(s)) = \vec{R}(s) = \begin{bmatrix} 4 \arcsin \sqrt{\frac{s}{8}} - \sin\left(4 \arcsin \sqrt{\frac{s}{8}}\right) \\ 1 - \cos\left(4 \arcsin \sqrt{\frac{s}{8}}\right) \end{bmatrix} \qquad 0 \le s \le 8$$

$$= \ldots =$$

<u>Curvature</u> is defined as the magnitude of the rate of change of the unit tangent vector of a curve wrt distance traveled.

unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

If parametrized by arclength, then $\|\vec{r}\,'(t)\|$ is 1

(other way: $s = \int^{t} \|\vec{r}\,'(t)\|\,dt$; differentiate and use FTC left side (if $s=t$) gives $\frac{ds}{dt} = 1$)

$$K = \|\vec{T}\,'(s)\| = \|\vec{r}\,''(s)\| \qquad \text{if } s \text{ is arclength}$$

By using the chain rule, we can express K also in terms of $\vec{r}(t)$ where t is any parameter

<u>Result of that calc:</u>

$$K(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

(can see calc. in book 13.4)

$$\text{speed} = v(t) = \frac{ds}{dt}$$

$$\vec{r}\,'(t) = v(t)\,\vec{T}(t)$$

$$\frac{d\vec{r}}{dt} = \frac{ds}{dt} \cdot \frac{d\vec{r}}{ds}$$

$$\vec{r}\,''(t) = v'(t)\,\vec{T}(t) + v(t)\,\vec{T}\,'(t)$$

calc $\vec{r}\,' \times \vec{r}\,''$ etc.

Rather than finishing up the calculation (see book!), let me interpret this formula a bit for you:

$\vec{r}\,'(t)$ is the velocity, $\vec{r}\,''(t)$ is the acceleration.

Acceleration can be decomposed into a tangential component (parallel to velocity $\vec{r}\,'$), which describes the change in speed, and a component orthogonal to $\vec{r}\,'$, which describes the change in direction.

In the cross product $\vec{r}\,' \times \vec{r}\,''$, only the component of $\vec{r}\,''$ that is orthog. to $\vec{r}\,'$, is relevant (b/c (parallel to $\vec{r}\,'$) $\times\, \vec{r}\,' = 0$). That's how it should be if the curvature doesn't depend on the parametrization.

Also you find three primes in the numerator: $\|\vec{r}\,' \times \vec{r}\,''\|$, and also three primes in the denominator: $\|\vec{r}\,'\|^3 = \|\vec{r}\,'\| \cdot \|\vec{r}\,'\| \cdot \|\vec{r}\,'\|$.

That's also crucial: If you reparametrize with, say, double speed, the inner derivative 2 pops out once for each prime, according to the chain rule; so these factors have to cancel if the curvature is to depend only on the curve, not on the speed with which it is traversed.