In a MV function f, if you freeze all but one variable and study the single variable fct of the remaining variable, then the single variable derivative of this latter function is called a partial derivative of f.

We denote partial deriv's with subscripts, rather than the prime of SVC fame b/c we need to be able to distinguish which variable is <u>not</u> frozen.

So $$\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

We use curly ∂ instead of the standard d for Leibniz notation of ~~SVC~~ SVC.

The wisdom of insisting on such distinction will transpire later.

(Preview: While the SV chain rule can be memorized as "cancelling" d terms as in $y(x(t))$: $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, no such cancellation rule is available for ∂ terms)

Example: $T(x,y)$ a temperature field
(temp' fct in dependence on 2 coord's (x,y))

$\frac{\partial T}{\partial x}$ is the rate of change of temp. wrt distance as you move eastward.

$\frac{\partial T}{\partial y}$ is the r.o.c. of temp. wrt ~~to~~ distance as you move northward.

—

Can calculate partial deriv's like in SVC.

$$\frac{\partial}{\partial x}\left(x^3y^2 + xy^4 - \sin(xy)\right) = 3x^2y^2 + y^4 - \cos(xy)\,y$$

$$\frac{\partial}{\partial y}\left(x^3y^2 + xy^4 - \sin(xy)\right) = x^3\cdot 2y + x\cdot 4y^3 - \cos(xy)\,x$$

—

Partial derivatives are directional derivatives in coordinate directions.

What are directional derivatives? (cf. Ch 14.5 after Example 5)

$\begin{bmatrix}1\\0\end{bmatrix}$ is a unit vector in x-direction

$\begin{bmatrix}0\\1\end{bmatrix}$ -"- y-direction

Let $\underbrace{\begin{bmatrix}h\\k\end{bmatrix}}_{=\vec{v}}$ be a vector in any direction (may choose a unit vector)

Then we define $D_{\vec{v}} f(x,y) = \lim_{t\to 0} \frac{f(x+th, y+tk) - f(x,y)}{t}$

$$= \frac{d}{dt} f(x+th, y+tk)\Big|_{t=0}$$

(if this limit exists)

$\vec{v}$ doesn't need to be a unit vector, but if $\|\vec{v}\|=1$, then the only extra info to be known about $\vec{v}$ is the direction.

We then call $D_{\vec{v}} f(x,y)$ the directional derivative of f at (x,y) in direction $\vec{v}$.

Vector notation: Replacing the point (x,y) with its position vector $\vec{x} = \begin{bmatrix}x\\y\end{bmatrix}$

we can write
$$D_{\vec{v}} f(\vec{x}) = \lim_{t\to 0} \frac{f(\vec{x}+t\vec{v}) - f(\vec{x})}{t} \quad (*)$$
$$= \frac{d}{dt} f(\vec{x}+t\vec{v})\Big|_{t=0}$$
without referring to components

For $\|\vec{v}\| = 1$, then

$D_{\vec{v}} f(\vec{x})$ is the rate of change of f wrt. distance at location $\vec{x}$ in direction $\vec{v}$

Definition $(*)$ makes sense even if $\|\vec{v}\| \neq 1$

Then $D_{\vec{v}} f(\vec{x})$ is the rate of change of f wrt. time at location $\vec{x}$ as you travel through domain of f with velocity $\vec{v}$.

<u>Example:</u> $f(x,y) = x^3y^2 + xy^4$

let $\vec{v} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$. Then $D_{\vec{v}} f(x,y) = \frac{d}{dt} f(x+\frac{3}{5}t, y+\frac{4}{5}t)\Big|_{t=0}$

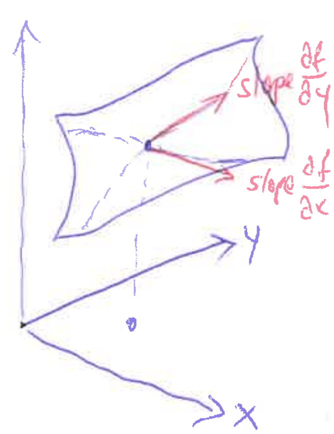
$$\frac{d}{dt}\left((x+\tfrac{3}{5}t)^3 (y+\tfrac{4}{5}t)^2 + (x+\tfrac{3}{5}t)(y+\tfrac{4}{5}t)^4 \right)\Big|_{t=0}$$

$= \ldots$ (calc as in SVC) $\ldots$

<u>Preview:</u> In many cases, directional deriv's can be re-assembled from partial derivatives. <u>But not always!</u>

<u>Imagine</u>: I know two slopes $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ of graph f and these are the known data that the surface graph f must obey.

Lots of different slopes are still possible as you go in intermediate directions!



So the green statement is actually quite surprising...