Example $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

asking about directional derivatives in direction $\vec{v} = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$ at $(0,0)$

$$\lim_{t\to 0} \frac{f(t\vec{v}) - f(\vec{0})}{t} = \lim_{t\to 0} \frac{t\cos\varphi\,(t\sin\varphi)^2}{(t^2\cos^2\varphi + t^2\sin^2\varphi)} \Big/ t = \lim_{t\to 0} \cos\varphi\,\sin^2\varphi$$

$$= \cos\varphi\,\sin^2\varphi$$

in direction $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (x direction $D_{\vec{v}} f = f_x$)

we have dir deriv. $= 0$

in direction $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad (D_{\vec{v}} f = f_y)$ $\quad D_{\vec{v}} f(0,0) = 0$

However in dir $\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ (45° direction) $\quad D_{\vec{v}} f(0,0) = \dfrac{\sqrt{2}}{4}$

If you plot these three direction vectors for tangent lines

$\underbrace{[1,0,}_{x\text{ dir.}}\underbrace{0]^T}_{\partial f/\partial x}$, $\quad \underbrace{[0,1,}_{y\text{ dir}}\underbrace{0]^T}_{\partial f/\partial y}$, $\quad \underbrace{\left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2},\right.}_{\text{NE dir.}} \underbrace{\left.\frac{\sqrt{2}}{4}\right]^T}_{\text{dir deriv in this dir.}}$

they do not assemble into a single plane.

in the origin,
The graph of f in this example will NOT have a plane that deserves to be called tangent plane

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If ~~a fi~~ the graph of a fct $f: (x,y) \mapsto f(x,y)$ has a tangent plane (ie a plane that deserves to be so called), then the dir. deriv's can be assembled from the partial derivatives & the direction vector (will explain later how)

A slightly trickier example:

$$f(x,y) = \begin{cases} \dfrac{x^4 y^2}{x^8+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

all the directional ~~deriv~~ derivatives at $(0,0)$ are 0

Nevertheless, the graph of this function does not have a tangent plane (ie a plane deserving to be called tangent) at $(0,0)$

Worse! This darn fct is not even continuous in the origin! How come? approach origin on parabola $y = x^2$

$$f(x, x^2) = \frac{x^4 (x^2)^2}{x^8 + (x^2)^4} = \frac{x^8}{2x^8} = \frac{1}{2}$$

(Tricky, I won't ask this on the exam, but it will help appreciate notion of tangent plane that is about to come)

Before we tackle tangent plane, a little leftover

Repeated partial derivatives

eg $f_{xy} = (f_x)_y$ (first do $\frac{\partial}{\partial x}$, then $\frac{\partial}{\partial y}$)

$\quad = \frac{\partial}{\partial y} \frac{\partial f}{\partial x}$

a nicer way of writing $\dfrac{\partial \frac{\partial f}{\partial x}}{\partial y}$

A question about order:

Is it true that $f_{xy} = f_{yx}$?

Answer: In most cases, yes; but there are exceptions.
$\downarrow$
won't show an example

Theorem: If f_{xy} and f_{yx} are both continuous fcts

then $f_{xy} = f_{yx}$