Differentiability (sec 14.4)

If any plane deserves to be called tangent plane to graph f, at (a,b), how would it look like?

It should contain the vectors $\begin{bmatrix} 1 \\ 0 \\ f_x(a,b) \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ f_y(a,b) \end{bmatrix}$

hence a normal vec to that plane would be the cross products of these

$\begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}$, and thus the plane would be

$$-f_x(a,b)\cdot x - f_y(a,b)\cdot y + 1\cdot z = \text{some const.}$$

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) \underbrace{+ \text{const} + \ldots}$$

may modify at the price of a new const

Clearly for the point $(a, b, f(a,b))$ to lie on the plane,

$$\boxed{z = f_x(a,b)\,(x-a) + f_y\,(y-b) + f(a,b)} =: L(x,y)$$

No other plane deserves to be called tangent plane, but does this one deserve the name?

As seen in prior examples, this is not ~~the~~ automatically the case

In SVC, diff'bility of f at a could be written as

$L(x)$ formula for tangent line

$$\cancel{\text{if}}\quad \frac{f(a+h) - \left(f(a) + \text{slope}\cdot h\right)}{h} \to 0 \text{ as } h \to 0$$

and the slope for which this works, will be $f'(a)$

Def: We call f differentiable at (a,b) if (f_x, f_y exist at (a,b) and)

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\left\| \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right\|} = 0$$

Thm: If f_x, f_y exist in a neighborhood of (a,b) and are continuous there, then f is differentiable there.

a little disc ~~[illegible]~~ around (a,b)
(with 3 var's : a little ball)

Directional derivatives:

If f is differentiable at (a,b), ~~then~~ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is a (unit) vector

then $D_{\vec{v}} f(a,b) = f_x(a,b) \cdot v_1 + f_y(a,b) \cdot v_2 = \underbrace{\begin{bmatrix} f_x(a,b) & f_y(a,b) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\text{matrix product}}$

don't worry if you haven't seen matrix products

How come?

$$D_{\vec{v}} f(a,b) = \lim_{t\to 0} \underbrace{\frac{f(a+tv_1, b+tv_2) - f(a,b)}{t}}$$

$$\frac{f(a+tv_1, b+tv_2) - [f(a,b) + f_x(a,b)tv_1 + f_y(a,b)tv_2]}{t} + f_x(a,b)v_1 + f_y(a,b)v_2$$

the $[\dots]$ is the $L(x,y)$

and the t in the denom. is $\pm \left\| \begin{bmatrix} a+tv_1 \\ b+tv_2 \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right\| = |t| \underbrace{\|\vec{v}\|}_{1}$

big fraction $\to 0$ b/c f is diff'ble

Linear Approximation

Recall an applic from SVC:

$$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}} \cdot 0.02 = 2.005$$

We approximate $f(x,y)$ in a small neighborhood of (a,b) by $L(x,y)$

$$f(a+\Delta x, b+\Delta y) \approx f(a,b) + f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

$$= \quad \text{—''—} \quad + \text{ error term}$$

~~approximate~~ change in f would be

$$\Delta f \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

When we work with the linear approximation, we call the approx. change (right side) df (and rename Δx, Δy into dx, dy respectively)

$$\boxed{df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy}$$ and we call

df <u>differential</u>